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DISCRETE CARLEMAN ESTIMATES FOR ELLIPTIC OPERATORS IN ARBITRARY DIMENSION AND APPLICATIONS*

FRANCK BOYER[†§], FLORENCE HUBERT[‡§], AND JÉRÔME LE ROUSSEAU[¶]

Abstract. In arbitrary dimension, we consider the semi-discrete elliptic operator $-\partial_t^2 + \mathcal{A}^{\mathfrak{M}}$, where $\mathcal{A}^{\mathfrak{M}}$ is a finite difference approximation of the operator $-\nabla_x(\Gamma(x)\nabla_x)$. For this operator we derive a global Carleman estimate, in which the usual large parameter is connected to the discretization step-size. We address discretizations on some families of smoothly varying meshes. We present consequences of this estimate such as a partial spectral inequality of the form of that proven by G. Lebeau and L. Robbiano for $\mathcal{A}^{\mathfrak{M}}$ and a null controllability result for the parabolic operator $\partial_t + \mathcal{A}^{\mathfrak{M}}$, for the lower part of the spectrum of $\mathcal{A}^{\mathfrak{M}}$. With the control function that we construct (whose norm is uniformly bounded) we prove that the L^2-norm of the final state converges to zero exponentially, as the step-size of the discretization goes to zero. A relaxed observability estimate is then deduced.

Key words. Elliptic operator – discrete and semi-discrete Carleman estimates – spectral inequality – control – parabolic equations.

AMS subject classifications. 35K05 - 65M06 - 93B05 - 93B07 - 93B40

1. Introduction and settings. Let $d \geq 2$, $L_1, \dots, L_d$ be positive real numbers, and $\Omega = \prod_{1\leq i\leq d} [0, L_i]$. We set $\boldsymbol{x} = (x_1, \dots, x_d) \in \Omega$. With $\omega \Subset \Omega$ we consider the following parabolic problem in $(0,T)\times\Omega$, with $T > 0$,

$$\partial_t y - \nabla_{\boldsymbol{x}} \cdot (\Gamma \nabla_{\boldsymbol{x}} y) = \mathbf{1}_\omega v \text{ in } (0,T)\times\Omega, \quad y_{|\partial\Omega} = 0, \quad \text{and } \; y_{|t=0} = y_0, \tag{1.1}$$

where the diagonal diffusion tensor $\Gamma(\boldsymbol{x}) = \mathrm{Diag}(\gamma_1(\boldsymbol{x}), \dots, \gamma_d(\boldsymbol{x}))$ with $\gamma_i(\boldsymbol{x}) > 0$ satisfies

$$\operatorname{reg}(\Gamma) \stackrel{\text{def}}{=} \sup_{\substack{\boldsymbol{x}\in\Omega\\ i=1,\dots,d}} \Big(\gamma_i(\boldsymbol{x}) + \frac{1}{\gamma_i(\boldsymbol{x})} + |\nabla_{\boldsymbol{x}}\gamma_i(\boldsymbol{x})|\Big) < +\infty. \tag{1.2}$$

The null-controllability problem consists in finding $v \in L^2((0,T)\times\Omega)$ such that $y(T) = 0$. This problem was solved in the 90's by G. Lebeau and L. Robbiano [LR95] and A. Fursikov and O. Yu. Imanuvilov [FI96].

Let us consider the elliptic operator on Ω given by

$$\mathcal{A} = -\nabla_{\boldsymbol{x}} \cdot (\Gamma \nabla_{\boldsymbol{x}}) = -\sum_{1\leq i\leq d} \partial_{x_i}(\gamma_i \partial_{x_i})$$

with homogeneous Dirichlet boundary conditions on $\partial\Omega$. We shall introduce a finite-difference approximation of the operator $\mathcal{A}$. For a mesh $\mathfrak{M}$ that we shall describe below, associated with a discretization step h, the discrete operator will be denoted

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by $\mathcal{A}^{\mathfrak{M}}$. It will act on a finite dimensional space $\mathbb{C}^{\mathfrak{M}}$, of dimension $|\mathfrak{M}|$, and will be selfadjoint for a suitable inner product in $\mathbb{C}^{\mathfrak{M}}$. Our main result is a Carleman-type estimate for the "extended" semi-discrete elliptic operator, $-\partial_t^2 + \mathcal{A}^{\mathfrak{M}}$. Here, the additional variable t is not directly connected to the time variable in the parabolic problem above. In the discrete setting, such a result was obtained in [BHL09a] in the one-dimensional case. Here, we extend this result to any space dimension. Note that we also prove a Carleman estimate for $\mathcal{A}^{\mathfrak{M}}$ itself. For Carleman estimates in the continuous case we refer to [Hör63, Zui83, Hör85, LR95, FI96, LR97, LL09]. Note that an earlier attempt at deriving discrete Carleman estimates can be found in [KS91]. The result presented in [KS91] cannot be used here as the condition imposed by these authors on the discretization step size, in connection to the large Carleman parameter, is too strong for the applications we have in mind to the problem of uniform controllability properties for semi-discrete parabolic problems.

We now describe an important consequence of the Carleman estimate we prove, which was the main motivation of this work. We denote by $\phi^{\mathfrak{M}}$ a set of discrete orthonormal eigenfunctions, $\phi_j \in \mathbb{C}^{\mathfrak{M}}$, $1 \leq j \leq |\mathfrak{M}|$, of the operator $\mathcal{A}^{\mathfrak{M}}$, and by $\mu^{\mathfrak{M}} = \{\mu_j, 1 \leq j \leq |\mathfrak{M}|\}$ the set of the associated eigenvalues sorted in a non-decreasing sequence. The following (partial) spectral inequality is then a corollary of the semi-discrete Carleman estimate we prove:

$$\sum_{\substack{\mu_k \in \mu^{\mathfrak{M}} \\ \mu_k \leq \mu}} |\alpha_k|^2 = \int_\Omega \Big| \sum_{\substack{\mu_k \in \mu^{\mathfrak{M}} \\ \mu_k \leq \mu}} \alpha_k \phi_k \Big|^2 \leq C e^{C\sqrt{\mu}} \int_\omega \Big| \sum_{\substack{\mu_k \in \mu^{\mathfrak{M}} \\ \mu_k \leq \mu}} \alpha_k \phi_k \Big|^2, \qquad \forall (\alpha_k)_{1 \leq k \leq |\mathfrak{M}|} \subset \mathbb{C}, \tag{1.3}$$

for $\mu h^2 \leq C_S$ with C_S and h sufficiently small (integrals of discrete functions are introduced below). This type of spectral inequality goes back to the work of G. Lebeau and L. Robbiano [LR95] (see also [LZ98a, JL99]). As opposed to the continuous case this inequality is not valid for the whole spectrum. The condition $\mu h^2 \leq C_S$ with C_S small, states that it is only valid for a constant lower portion of the spectrum. This condition cannot be relaxed. The optimal value of C_S is not known at this point and certainly depends, at least, on the geometry of ω.

The spectral inequality (1.3) then implies the null-controllability of system (1.1) for the lower part of the spectrum $\mu \leq C_S/h^2$, *i.e.*, for any initial condition $y_0 \in \mathbb{C}^{\mathfrak{M}}$, there exists a control v in $L^2((0,T) \times \Omega)$ (the semi-discrete functional spaces we shall use will be made precise below) with $\|v\|_{L^2((0,T)\times\Omega)} \leq C|y_0|_{L^2(\Omega)}$ such that $(y(T), \phi_k) = 0$ if $\mu_k h^2 \leq C_S$. Moreover, the remainder satisfies $|y(T)|_{L^2(\Omega)} \leq e^{-C/h^2}|y_0|_{L^2(\Omega)}$. We thus obtain an exponential convergence as h goes to 0. Accurate statements of the results we have just described are given in Section 1.2.

The form of the relaxed observability estimate that follows from this controllability result has been the inspiration for the study of Carleman estimates for semi-discrete parabolic operators [BHL10]. The spectral inequality (1.3) is also at the heart of the work carried out by the authors on the numerical analysis of the fully-discretized parabolic control problem in [BHL09b].

In two dimensions, for finite differences, there is a counterexample to the null and approximate controllabilities for uniform grids on a square domain for distributed or boundary controls due to O. Kavian (see [Zua06]). It exploits an explicit eigenfunction of $\mathcal{A}^{\mathfrak{M}}$ in two dimensions that is solely localized on the diagonal of the square domain. This eigenfunction is associated with an eigenvalue in the higher part of the spectrum. Our result may thus seem rather optimal in dimension greater that two.

In dimension one, there is a null controllability result due to A. Lopez and E. Zuazua [LZ98b] for the *entire* spectrum in the case of a *constant* diffusion coefficient and for a *constant step size* finite-difference discretization. In dimension one, our method based on the proof of discrete Lebeau-Robbiano spectral inequality cannot achieve such a result. In fact, one can notice that (1.3) cannot hold for the full spectrum. In dimension one, the generalization of the result of [LZ98b] to a non constant coefficient and non uniform meshes remains an open problem.

We now present the precise settings we shall work with.

For $1 \leq i \leq d$, $i \in \mathbb{N}$, we set $\Omega_i = \prod_{\substack{1\leq j\leq d\\ j\neq i}} [0, L_j]$. For $\mathcal{T} > 0$ we introduce

$$Q = (0, \mathcal{T}) \times \Omega, \quad Q_i = (0, \mathcal{T}) \times \Omega_i, \quad 1 \leq i \leq d.$$

We also set boundaries as (see Figure 1)

$$\partial_i^- \Omega = \prod_{1\leq j<i} [0, L_j] \times \{0\} \times \prod_{i<j\leq d} [0, L_j], \quad \partial_i^+ \Omega = \prod_{1\leq j<i} [0, L_j] \times \{L_i\} \times \prod_{i<j\leq d} [0, L_j],$$
$$\partial_i \Omega = \partial_i^+ \Omega \cup \partial_i^- \Omega, \quad \partial\Omega = \bigcup_{1\leq i\leq d} \partial_i \Omega.$$

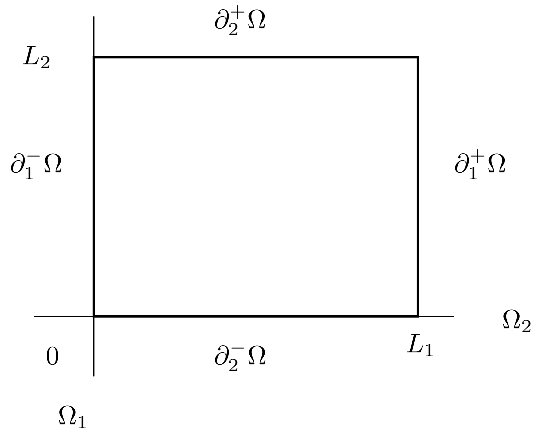

FIG. 1. *Notation for the boundaries*

1.1. Discrete settings. Here, we precisely define the type of mesh and discretization we shall use. The notation we introduce is technical, and yet will allow us to use a formalism as close as possible to the continuous case, in particular for norms and integrations. *Then most of the computations we carry out can be read in a very intuitive manner, which will ease the reading of the article.* Most of the discrete formalism will then be hidden in the subsequent sections. The notation below is however necessary for a complete and precise reading of the proofs.

We shall use the notation $[\![a, b]\!] = [a, b] \cap \mathbb{N}$.

1.1.1. Primal mesh. For $i \in [\![1, d]\!]$ and $N_i \in \mathbb{N}^*$, let

$$0 = x_{i,0} < x_{i,1} < \cdots < x_{i,N_i} < x_{i,N_i+1} = L_i.$$

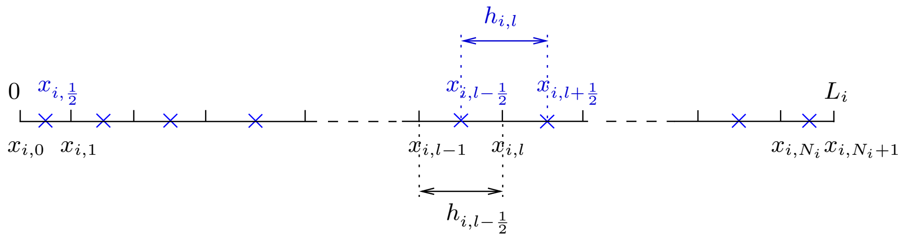

FIG. 2. *Discretization in the ith direction.*

We introduce the following set of indices,

$$\mathfrak{N} := \big\{ \boldsymbol{k} = (k_1, \dots, k_d);\ \ k_i \in [\![1, N_i]\!],\ i \in [\![1, d]\!] \big\}.$$

For $\boldsymbol{k} = (k_1, \dots, k_d) \in \mathfrak{N}$ we set $\boldsymbol{x_k} = (x_{1,k_1}, \dots, x_{d,k_d}) \in \Omega$. We refer to this discretization as to the primal mesh

$$\mathfrak{M} := \big\{ \boldsymbol{x_k};\ \boldsymbol{k} \in \mathfrak{N} \big\}, \quad \text{with } |\mathfrak{M}| := \prod_{i \in [\![1,d]\!]} N_i.$$

For $i \in [\![1, d]\!]$ and $l \in [\![0, N_i]\!]$ we set

$$h_{i,l+\frac{1}{2}} = x_{i,l+1} - x_{i,l}, \qquad x_{i,l+\frac{1}{2}} = (x_{i,l+1} + x_{i,l})/2,$$

and

$$h_i = \max_{l \in [\![0,N_i]\!]} h_{i,l+\frac{1}{2}}, \quad i \in [\![1, d]\!], \qquad h = \max_{i \in [\![1,d]\!]} h_i.$$

For $i \in [\![1, d]\!]$ and $l \in [\![1, N_i]\!]$, we set

$$h_{i,l} = x_{i,l+\frac{1}{2}} - x_{i,l-\frac{1}{2}} = (h_{i,l+\frac{1}{2}} + h_{i,l-\frac{1}{2}})/2.$$

See Figure 2, where the introduced notation is illustrated.

1.1.2. Boundary of the primal mesh. To introduce boundary conditions in the ith direction and related trace operators (see Section 1.1.5) we set $\partial_i \mathfrak{N} = \partial_i^- \mathfrak{N} \cup \partial_i^+ \mathfrak{N}$ with

$$\begin{aligned} \partial_i^- \mathfrak{N} &= \big\{ \boldsymbol{k} = (k_1, \dots, k_d);\ k_j \in [\![1, N_j]\!],\ j \in [\![1, d]\!],\ j \neq i,\ k_i = 0 \big\}, \\ \partial_i^+ \mathfrak{N} &= \big\{ \boldsymbol{k} = (k_1, \dots, k_d);\ k_j \in [\![1, N_j]\!],\ j \in [\![1, d]\!],\ j \neq i,\ k_i = N_i + 1 \big\}, \end{aligned}$$

and

$$\partial \mathfrak{N} = \bigcup_{i \in [\![1,d]\!]} \partial_i \mathfrak{N}, \qquad \partial \mathfrak{M} = \big\{ \boldsymbol{x_k};\ \boldsymbol{k} \in \partial \mathfrak{N} \big\}, \quad \partial_i^{\pm} \mathfrak{M} = \big\{ \boldsymbol{x_k};\ \boldsymbol{k} \in \partial_i^{\pm} \mathfrak{N} \big\}.$$

Notice that $\partial_i^{\pm} \mathfrak{M}$ is nothing but the set of points of the primal mesh which are located on the boundary $\partial_i^{\pm} \Omega$.

1.1.3. Dual meshes. We will need to operate discrete derivatives on functions defined on the primal mesh (see Section 1.1.6). It is easily seen that these derivatives are naturally associated to another set of meshes, called dual meshes. In fact there will be two kinds of such meshes: the ones associated to first order discrete derivation and the ones associated to second order discrete derivation. Let us define precisely these new meshes.

For $i \in [\![1,d]\!]$, we introduce a second type of sets of indices

$$\overline{\mathfrak{N}}^{i} := \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_j \in [\![1,N_j]\!]\ j \in [\![1,d]\!],\ j \neq i,\\ \text{and } k_i = l + \frac{1}{2},\ l \in [\![0,N_i]\!]\Big\}.$$

For $j \in [\![1,d]\!]$, $j \neq i$, we also set $\partial_j \overline{\mathfrak{N}}^{i} = \partial_j^- \overline{\mathfrak{N}}^{i} \cup \partial_j^+ \overline{\mathfrak{N}}^{i}$ with

$$\partial_j^- \overline{\mathfrak{N}}^{i} = \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_{i'} \in [\![1,N_{i'}]\!],\ i' \in [\![1,d]\!],\ i' \neq i,\ i' \neq j,\\ k_i = l + \frac{1}{2},\ l \in [\![0,N_i]\!],\ \text{and } k_j = 0\Big\},$$

$$\partial_j^+ \overline{\mathfrak{N}}^{i} = \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_{i'} \in [\![1,N_{i'}]\!],\ i' \in [\![1,d]\!],\ i' \neq i,\ i' \neq j,\\ k_i = l + \frac{1}{2},\ l \in [\![0,N_i]\!],\ \text{and } k_j = N_j + 1\Big\},$$

and $\partial \overline{\mathfrak{N}}^{i} = \cup_{\substack{j \in [\![1,d]\!] \\ j \neq i}} \partial_j \overline{\mathfrak{N}}^{i}$. We moreover introduce $\partial_i \overline{\mathfrak{N}}^{i} = \partial_i^- \overline{\mathfrak{N}}^{i} \cup \partial_i^+ \overline{\mathfrak{N}}^{i}$ with

$$\partial_i^- \overline{\mathfrak{N}}^{i} = \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_j \in [\![1,N_j]\!],\ j \in [\![1,d]\!],\ j \neq i,\ k_i = \frac{1}{2}\Big\},$$

$$\partial_i^+ \overline{\mathfrak{N}}^{i} = \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_j \in [\![1,N_j]\!],\ j \in [\![1,d]\!],\ j \neq i,\ k_i = N_i + \frac{1}{2}\Big\}.$$

Remark that $\partial_i \overline{\mathfrak{N}}^{i} \subset \overline{\mathfrak{N}}^{i}$ whereas $\partial_j \overline{\mathfrak{N}}^{i} \not\subset \overline{\mathfrak{N}}^{i}$ for $j \neq i$.

For $i, j \in [\![1,d]\!]$, $i \neq j$, we introduce a third type of sets of indices

$$\overline{\mathfrak{N}}^{ij} := \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_{i'} \in [\![1,N_{i'}]\!],\ i' \in [\![1,d]\!],\ i' \neq i,\quad i' \neq j\\ \text{and } k_i = l_1 + \frac{1}{2},\ l_1 \in [\![0,N_i]\!],\ k_j = l_2 + \frac{1}{2},\ l_2 \in [\![0,N_j]\!],\Big\}.$$

For $l \in [\![1,d]\!]$, $l \neq i$, $l \neq j$, we also set $\partial_l \overline{\mathfrak{N}}^{ij} = \partial_l^- \overline{\mathfrak{N}}^{ij} \cup \partial_l^+ \overline{\mathfrak{N}}^{ij}$ with

$$\partial_l^- \overline{\mathfrak{N}}^{ij} = \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_{i'} \in [\![1,N_{i'}]\!],\ i' \in [\![1,d]\!],\ i' \neq i,\ i' \neq j,\ i' \neq l,\\ k_i = l_1 + \frac{1}{2},\ l_1 \in [\![0,N_i]\!],\ k_j = l_2 + \frac{1}{2},\ l_2 \in [\![0,N_j]\!],\ \text{and } k_l = 0\Big\},$$

$$\partial_l^+ \overline{\mathfrak{N}}^{ij} = \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_{i'} \in [\![1,N_{i'}]\!],\ i' \in [\![1,d]\!],\ i' \neq i,\ i' \neq j,\ i' \neq l,\\ k_i = l_1 + \frac{1}{2},\ l_1 \in [\![0,N_i]\!],\ k_j = l_2 + \frac{1}{2},\ l_2 \in [\![0,N_j]\!],\ \text{and } k_l = N_l + 1\Big\},$$

and $\partial \overline{\mathfrak{N}}^{ij} = \cup_{\substack{l \in [\![1,d]\!] \\ l \neq i, l \neq j}} \partial_l \overline{\mathfrak{N}}^{ij}$. Moreover we set $\partial_i \overline{\mathfrak{N}}^{ij} = \partial_i^- \overline{\mathfrak{N}}^{ij} \cup \partial_i^+ \overline{\mathfrak{N}}^{ij}$ with

$$\partial_i^- \overline{\mathfrak{N}}^{ij} = \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_{i'} \in [\![1,N_{i'}]\!],\ i' \in [\![1,d]\!],\ i' \neq i,\ i' \neq j,\\ k_i = \frac{1}{2},\ k_j = l + \frac{1}{2},\ l \in [\![0,N_j]\!]\Big\},$$

$$\partial_i^+ \overline{\mathfrak{N}}^{ij} = \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_{i'} \in [\![1,N_{i'}]\!],\ i' \in [\![1,d]\!],\ i' \neq i,\ i' \neq j,\\ k_i = N_i + \frac{1}{2},\ k_j = l + \frac{1}{2},\ l \in [\![0,N_j]\!]\Big\}.$$

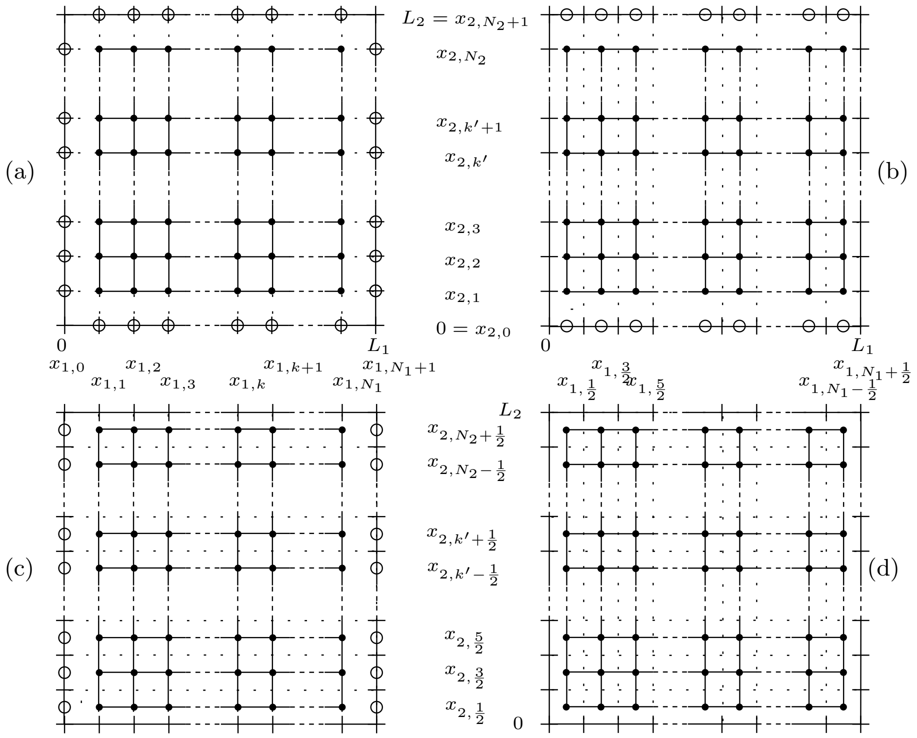

FIG. 3. *Primal mesh and dual meshes in the two-dimensional case. The mesh points are marked by black discs. Available boundary mesh points are marked with white discs.* (a) $\mathfrak{M}$ *and* $\partial\mathfrak{M}$; (b) $\overline{\mathfrak{M}}^1$ *and* $\partial\overline{\mathfrak{M}}^1$; (c) $\overline{\mathfrak{M}}^2$ *and* $\partial\overline{\mathfrak{M}}^2$; (d) $\overline{\mathfrak{M}}^{12}$.

For $\boldsymbol{k} = (k_1, \dots, k_d) \in \overline{\mathfrak{N}}^i$ or $\partial\overline{\mathfrak{N}}^i$ (resp. $\overline{\mathfrak{N}}^{ij}$ or $\partial\overline{\mathfrak{N}}^{ij}$) we also set $\boldsymbol{x_k} = (x_{1,k_1}, \dots, x_{d,k_d})$, which gives the following dual meshes

$$\overline{\mathfrak{M}}^i := \{\boldsymbol{x_k};\ \boldsymbol{k} \in \overline{\mathfrak{N}}^i\}, \quad \partial\overline{\mathfrak{M}}^i := \{\boldsymbol{x_k};\ \boldsymbol{k} \in \partial\overline{\mathfrak{N}}^i\}, \quad \partial_j^{\pm}\overline{\mathfrak{M}}^i := \{\boldsymbol{x_k};\ \boldsymbol{k} \in \partial_j^{\pm}\overline{\mathfrak{N}}^i\},$$
$$\big(\text{resp. } \overline{\mathfrak{M}}^{ij} := \{\boldsymbol{x_k};\ \boldsymbol{k} \in \overline{\mathfrak{N}}^{ij}\}, \quad \partial\overline{\mathfrak{M}}^{ij} := \{\boldsymbol{x_k};\ \boldsymbol{k} \in \partial\overline{\mathfrak{N}}^{ij}\},$$
$$\partial_l^{\pm}\overline{\mathfrak{M}}^{ij} := \{\boldsymbol{x_k};\ \boldsymbol{k} \in \partial_l^{\pm}\overline{\mathfrak{N}}^{ij}\}\big).$$

The geometry of the different meshes we have introduced is illustrated in Figure 2 in the two dimensional case.

In the present article, we shall only consider some families of regular non uniform meshes, that will be precisely defined in Section 1.1.8. Note that the extension of our results to more general mesh families does not seem to be straightforward.

1.1.4. Discrete functions. We denote by $\mathbb{C}^{\mathfrak{M}}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^i}$ or $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}}$) the sets of discrete functions defined on $\mathfrak{M}$ (resp. $\overline{\mathfrak{M}}^i$ or $\overline{\mathfrak{M}}^{ij}$) respectively. If $u \in \mathbb{C}^{\mathfrak{M}}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^i}$ or $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}}$), we denote by $u_{\boldsymbol{k}}$ its value corresponding to $\boldsymbol{x_k}$ for $\boldsymbol{k} \in \mathfrak{N}$ (resp. $\boldsymbol{k} \in \overline{\mathfrak{N}}^i$ or $\boldsymbol{k} \in \overline{\mathfrak{N}}^{ij}$). For $u \in \mathbb{C}^{\mathfrak{M}}$ we define

$$u^{\mathfrak{M}} = \sum_{\boldsymbol{k} \in \mathfrak{N}} \mathbf{1}_{b_{\boldsymbol{k}}}\, u_{\boldsymbol{k}} \in L^\infty(\Omega), \quad \text{with } b_{\boldsymbol{k}} = \prod_{i \in [\![1,d]\!]} [x_{i,k_i-\frac{1}{2}}, x_{i,k_i+\frac{1}{2}}],\ \boldsymbol{k} \in \mathfrak{N}. \tag{1.4}$$

Since no confusion is possible, by abuse of notation we shall often write u in place of $u^{\mathfrak{M}}$. For $u \in \mathbb{C}^{\mathfrak{M}}$ we define

$$\iint_\Omega u := \iint_\Omega u^{\mathfrak{M}}(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{\boldsymbol{k}\in\mathfrak{N}} \big|b_{\boldsymbol{k}}\big|\, u_{\boldsymbol{k}}, \quad \text{where } \big|b_{\boldsymbol{k}}\big| = \prod_{i\in[\![1,d]\!]} h_{i,k_i},\ \boldsymbol{k}\in\mathfrak{N}.$$

For some $u \in \mathbb{C}^{\mathfrak{M}}$, we shall need to associate boundary values

$$u^{\partial\mathfrak{M}} = \big\{u_{\boldsymbol{k}};\ \boldsymbol{k}\in\partial\mathfrak{N}\big\},$$

i.e., the values of u at the point $\boldsymbol{x_k} \in \partial\mathfrak{M}$. The set of such extended discrete functions is denoted by $\mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$. Homogeneous Dirichlet boundary conditions then consist in the choice $u_{\boldsymbol{k}} = 0$ for $\boldsymbol{k} \in \partial\mathfrak{N}$, in short $u^{\partial\mathfrak{M}} = 0$ or even $u_{|\partial\Omega} = 0$ by abuse of notation (see also Section 1.1.5 below).

Similarly, for $u \in \mathbb{C}^{\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}}$) we shall associate the following boundary values

$$u^{\partial\overline{\mathfrak{M}}^i} = \big\{u_{\boldsymbol{k}};\ \boldsymbol{k}\in\partial\overline{\mathfrak{N}}^i\big\}\ \ \big(\text{resp. } u^{\partial\overline{\mathfrak{M}}^{ij}} = \big\{u_{\boldsymbol{k}};\ \boldsymbol{k}\in\partial\overline{\mathfrak{N}}^{ij}\big\}\big).$$

The set of such extended discrete functions is denoted by $\mathbb{C}^{\overline{\mathfrak{M}}^i\cup\partial\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}\cup\partial\overline{\mathfrak{M}}^{ij}}$).

For $u \in \mathbb{C}^{\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}}$) we define

$$u^{\overline{\mathfrak{M}}^i} = \sum_{\boldsymbol{k}\in\overline{\mathfrak{N}}^i} \mathbf{1}_{\overline{b}^i_{\boldsymbol{k}}} u_{\boldsymbol{k}} \in L^\infty(\Omega) \quad \text{with } \overline{b}^i_{\boldsymbol{k}} = \prod_{l\in[\![1,d]\!]} [x_{l,k_l-\frac12}, x_{l,k_l+\frac12}],\ \boldsymbol{k}\in\overline{\mathfrak{N}}^i,$$

$$\Big(\text{resp.} u^{\overline{\mathfrak{M}}^{ij}} = \sum_{\boldsymbol{k}\in\overline{\mathfrak{N}}^{ij}} \mathbf{1}_{\overline{b}^{ij}_{\boldsymbol{k}}} u_{\boldsymbol{k}} \in L^\infty(\Omega) \quad \text{with } \overline{b}^{ij}_{\boldsymbol{k}} = \prod_{l\in[\![1,d]\!]} [x_{l,k_l-\frac12}, x_{l,k_l+\frac12}],\ \boldsymbol{k}\in\overline{\mathfrak{N}}^{ij}\Big).$$

As above, for $u \in \mathbb{C}^{\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}}$), we define

$$\iint_\Omega u := \iint_\Omega u^{\overline{\mathfrak{M}}^i}(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{\boldsymbol{k}\in\overline{\mathfrak{N}}^i} \big|\overline{b}^i_{\boldsymbol{k}}\big|\, u_{\boldsymbol{k}}, \quad \text{where } \big|\overline{b}^i_{\boldsymbol{k}}\big| = \prod_{l\in[\![1,d]\!]} h_{l,k_l},\ \boldsymbol{k}\in\overline{\mathfrak{N}}^i,$$

$$\Big(\text{resp.} \iint_\Omega u := \iint_\Omega u^{\overline{\mathfrak{M}}^{ij}}(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{\boldsymbol{k}\in\overline{\mathfrak{N}}^{ij}} \big|\overline{b}^{ij}_{\boldsymbol{k}}\big|\, u_{\boldsymbol{k}}, \quad \text{where } \big|\overline{b}^{ij}_{\boldsymbol{k}}\big| = \prod_{l\in[\![1,d]\!]} h_{l,k_l},\ \boldsymbol{k}\in\overline{\mathfrak{N}}^{ij}\Big).$$

REMARK 1.1. *Above, the definitions of $b_{\boldsymbol{k}}$, $\overline{b}^i_{\boldsymbol{k}}$, and $\overline{b}^{ij}_{\boldsymbol{k}}$ look similar. They are however different as each time the multi-index $\boldsymbol{k} = (k_1, \dots, k_d)$ is chosen in a different set: $\mathfrak{N}$, $\overline{\mathfrak{N}}^i$ and $\overline{\mathfrak{N}}^{ij}$ respectively.*

With $u(t)$ in $\mathbb{C}^{\mathfrak{M}}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^i}$ or $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}}$) for all $t \in (0, \mathcal{T})$, we shall write $\iiint_Q u\, dt = \int_0^{\mathcal{T}}\iint_\Omega u(t)\, dt$. In particular we define the following L^2 inner product on $\mathbb{C}^{\mathfrak{M}}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^i}$ or $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}}$)

$$(u,v)_{L^2(\Omega)} = \iint_\Omega uv^* = \iint_\Omega u^{\mathfrak{M}}(\boldsymbol{x})(v^{\mathfrak{M}}(\boldsymbol{x}))^*\, d\boldsymbol{x}, \tag{1.5}$$

$$\Big(\text{resp. } (u,v)_{L^2(\Omega)} = \iint_\Omega uv^* = \iint_\Omega u^{\overline{\mathfrak{M}}^i}(\boldsymbol{x})(v^{\overline{\mathfrak{M}}^i}(\boldsymbol{x}))^*\, d\boldsymbol{x},$$

$$\text{or } (u,v)_{L^2(\Omega)} = \iint_\Omega uv^* = \iint_\Omega u^{\overline{\mathfrak{M}}^{ij}}(\boldsymbol{x})(v^{\overline{\mathfrak{M}}^{ij}}(\boldsymbol{x}))^*\, d\boldsymbol{x}\Big).$$

The associated norms will be denoted by $|u|_{L^2(\Omega)}$. For semi-discrete function $u(t)$, $t \in (0, \mathcal{T})$, as above we shall also use the following L^2 norm:

$$\|u(t)\|^2_{L^2(Q)} = \int_0^{\mathcal{T}}\iint_\Omega |u(t)|^2 dt.$$

1.1.5. Traces. Let $i \in [\![1, d]\!]$. For $u \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^j\cup\partial\overline{\mathfrak{M}}^j}$, $j \neq i$), its trace on $\partial_i^+\Omega$, corresponds to $\boldsymbol{k} \in \partial_i^+\mathfrak{N}$ (resp. $\partial_i^+\overline{\mathfrak{N}}^j$), *i.e.*, $k_i = N_i + 1$ in our discretization and will be denoted by $u_{|k_i=N_i+1}$ or simply u_{N_i+1}. Similarly its trace on $\partial_i^-\Omega$, corresponds to $\boldsymbol{k} \in \partial_i^-\mathfrak{N}$ (resp. $\partial_i^-\overline{\mathfrak{N}}^j$), *i.e.*, $k_i = 0$ and will be denoted by $u_{|k_i=0}$ or simply u_0. The latter notation will be used if no confusion is possible, if the context indicates that the trace is taken on $\partial_i^-\Omega$.

By abuse of notation, we shall also use $\partial_i\Omega$, $i \in [\![1, d]\!]$, to denote the boundaries of Ω in the discrete setting. For homogeneous Dirichlet boundary condition we shall write

$$v_{|\partial_i\Omega} = 0 \quad \Leftrightarrow \quad v_{|k_i=0} = v_{|k_i=N_i+1} = 0.$$

For $v \in \mathbb{C}^{\overline{\mathfrak{M}}^i\cup\partial\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}\cup\partial\overline{\mathfrak{M}}^{ij}}$, $j \neq i$), its trace on $\partial_i^+\Omega$, corresponds to $\boldsymbol{k} \in \partial_i^+\overline{\mathfrak{N}}^i$ (resp. $\partial_i^+\overline{\mathfrak{N}}^{ij}$), *i.e.*, $k_i = N_i + \frac{1}{2}$ in our discretization and will be denoted by $v_{|k_i=N_i+\frac12}$ or simply $v_{N_i+\frac12}$. Similarly its trace on $\partial_i^-\Omega$, corresponds to $\boldsymbol{k} \in \partial_i^-\overline{\mathfrak{N}}^i$ (resp. $\partial_i^-\overline{\mathfrak{N}}^{ij}$), *i.e.*, $k_i = \frac12$ and will be denoted by $v_{|k_i=\frac12}$ or simply $v_{\frac12}$. The latter notation will be used if no confusion is possible, if the context indicates that the trace is taken on $\partial_i^-\Omega$.

For such functions $u \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^j\cup\partial\overline{\mathfrak{M}}^j}$, $j \neq i$) we can then define surface integrals of the type

$$\int_{\partial_i^+\Omega} u_{|\partial_i^+\Omega} = \int_{\Omega_i} u_{|k_i=N_i+1} = \sum_{\substack{\boldsymbol{k}\in\partial_i^+\mathfrak{N} \\ (\text{resp. } \boldsymbol{k}\in\partial_i^+\overline{\mathfrak{N}}^j)}} \left|\partial_i b_{\boldsymbol{k}}\right| u_{\boldsymbol{k}},$$
$$\text{where } \left|\partial_i b_{\boldsymbol{k}}\right| = \prod_{\substack{l\in[\![1,d]\!] \\ l\neq i}} h_{l,k_l}, \ \boldsymbol{k} \in \partial_i^+\mathfrak{N} \text{ (resp. } \partial_i^+\overline{\mathfrak{N}}^j),$$

and for $v \in \mathbb{C}^{\overline{\mathfrak{M}}^i\cup\partial\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}\cup\partial\overline{\mathfrak{M}}^{ij}}$, $j \neq i$)

$$\int_{\partial_i^+\Omega} v_{|\partial_i^+\Omega} = \int_{\Omega_i} v_{|k_i=N_i+\frac12} = \sum_{\substack{\boldsymbol{k}\in\partial_i^+\overline{\mathfrak{N}}^i \\ (\text{resp. } \boldsymbol{k}\in\partial_i^+\overline{\mathfrak{N}}^{ij})}} \left|\partial_i \overline{b}^i_{\boldsymbol{k}}\right| v_{\boldsymbol{k}},$$
$$\text{where } \left|\partial_i \overline{b}^i_{\boldsymbol{k}}\right| = \prod_{\substack{l\in[\![1,d]\!] \\ l\neq i}} h_{l,k_l}, \ \boldsymbol{k} \in \partial_i^+\overline{\mathfrak{N}}^i \text{ (resp. } \partial_i^+\overline{\mathfrak{N}}^{ij}).$$

Observe that if $\boldsymbol{k} \in \partial_i^+\mathfrak{N}$ (resp. $\partial_i^+\overline{\mathfrak{N}}^j$) and $\boldsymbol{k}' \in \partial_i^+\overline{\mathfrak{N}}^i$ (resp. $\partial_i^+\overline{\mathfrak{N}}^{ij}$) with $k_l = k'_l$ for $l \neq i$ then $\left|\partial_i b_{\boldsymbol{k}}\right| = \left|\partial_i \overline{b}^i_{\boldsymbol{k}'}\right|$. We thus have

$$\int_{\partial_i^+\Omega} v_{|\partial_i^+\Omega} = \int_{\Omega_i} v_{|k_i=N_i+\frac12} = \int_{\Omega_i} (\tau_i^- v)_{|k_i=N_i+1} = \int_{\partial_i^+\Omega} (\tau_i^- v)_{|\partial_i^+\Omega}$$

where $\tau_i^- v \in \mathbb{C}^{\mathfrak{M}\cup\partial_i^-\mathfrak{M}}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^j\cup\partial_i^-\overline{\mathfrak{M}}^j}$) with the translation operator τ_i^- defined in Section 1.1.6. It is then natural to define the following integrals

$$\int_{\Omega_i} u_{N_i+1} v_{N_i+\frac12} = \int_{\Omega_i} u_{|k_i=N_i+1} v_{|k_i=N_i+\frac12} = \int_{\Omega_i} (u\,\tau_i^- v)_{|k_i=N_i+1} = \int_{\partial_i^+\Omega} u(\tau_i^- v)_{|\partial_i^+\Omega}.$$

Such trace integrals will appear when applying discrete integrations by parts in the following sections.

Similar definitions and considerations can be made for integrals over $\partial_i^-\Omega$.

For $u \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^j\cup\partial\overline{\mathfrak{M}}^j}$, $j \neq i$) we can then introduce the following L^2 norm for the trace on $\partial_i\Omega$:

$$|u|^2_{L^2(\partial_i\Omega)} = |u_{|\partial_i\Omega}|^2_{L^2(\partial_i\Omega)} = \int_{\Omega_i} \left|u_{|k_i=N_i+1}\right|^2 + \int_{\Omega_i} \left|u_{|k_i=0}\right|^2.$$

For $v \in \mathbb{C}^{\overline{\mathfrak{M}}^i\cup\partial\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}\cup\partial\overline{\mathfrak{M}}^{ij}}$, $j \neq i$) we can then introduce the following L^2 norm for the trace on $\partial_i\Omega$:

$$|v|^2_{L^2(\partial_i\Omega)} = |v_{|\partial_i\Omega}|^2_{L^2(\partial_i\Omega)} = \int_{\Omega_i} \left|u_{|k_i=N_i+\frac{1}{2}}\right|^2 + \int_{\Omega_i} \left|u_{|k_i=\frac{1}{2}}\right|^2.$$

1.1.6. Difference operators. Let $i, j \in [\![1, d]\!]$, $j \neq i$. We define the following translations for indices:

$$\begin{gathered}\tau_i^{\pm} : \overline{\mathfrak{N}}^i \text{ (resp. } \overline{\mathfrak{N}}^{ij}) \to \mathfrak{N} \cup \partial_i^{\pm}\mathfrak{N} \text{ (resp. } \overline{\mathfrak{N}}^j \cup \partial_i^{\pm}\overline{\mathfrak{N}}^j),\\ \boldsymbol{k} \mapsto \tau_i^{\pm}\boldsymbol{k},\end{gathered}$$

with

$$(\tau_i^{\pm}\boldsymbol{k})_l = \begin{cases} k_l & \text{if } l \neq i,\\ k_l \pm \frac{1}{2} & \text{if } l = i.\end{cases}$$

Translations operators mapping $\mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}} \to \mathbb{C}^{\overline{\mathfrak{M}}^i}$ and $\mathbb{C}^{\overline{\mathfrak{M}}^j\cup\partial\overline{\mathfrak{M}}^j} \to \mathbb{C}^{\overline{\mathfrak{M}}^{ij}}$ are then given by

$$(\tau_i^{\pm}u)_{\boldsymbol{k}} = u_{(\tau_i^{\pm}\boldsymbol{k})}, \quad \boldsymbol{k} \in \overline{\mathfrak{N}}^i \text{ (resp. } \overline{\mathfrak{N}}^{ij}).$$

A difference operator D_i and an averaging operator A_i are then given by

$$\begin{aligned}(D_i u)_{\boldsymbol{k}} &= (h_{i,k_i})^{-1}((\tau_i^+u)_{\boldsymbol{k}} - (\tau_i^-u)_{\boldsymbol{k}}), \quad \boldsymbol{k} \in \overline{\mathfrak{N}}^i \text{ (resp. } \overline{\mathfrak{N}}^{ij}),\\ (A_i u)_{\boldsymbol{k}} &= \tilde{u}^i_{\boldsymbol{k}} = \frac{1}{2}((\tau_i^+u)_{\boldsymbol{k}} + (\tau_i^-u)_{\boldsymbol{k}}), \quad \boldsymbol{k} \in \overline{\mathfrak{N}}^i \text{ (resp. } \overline{\mathfrak{N}}^{ij}).\end{aligned}$$

Both map $\mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}} \to \mathbb{C}^{\overline{\mathfrak{M}}^i}$ and $\mathbb{C}^{\overline{\mathfrak{M}}^j\cup\partial\overline{\mathfrak{M}}^j} \to \mathbb{C}^{\overline{\mathfrak{M}}^{ij}}$.

We also define the following translations for indices:

$$\begin{gathered}\tau_i^{\pm} : \mathfrak{N} \text{ (resp. } \overline{\mathfrak{N}}^j) \to \overline{\mathfrak{N}}^i \text{ (resp. } \overline{\mathfrak{N}}^{ij}),\\ \boldsymbol{k} \mapsto \tau_i^{\pm}\boldsymbol{k},\end{gathered}$$

with

$$(\tau_i^{\pm}\boldsymbol{k})_l = \begin{cases} k_l & \text{if } l \neq i,\\ k_l \pm \frac{1}{2} & \text{if } l = i.\end{cases}$$

Translations operators mapping $\mathbb{C}^{\overline{\mathfrak{M}}^i} \to \mathbb{C}^{\mathfrak{M}}$ and $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}} \to \mathbb{C}^{\overline{\mathfrak{M}}^j}$ are then given by

$$(\tau_i^{\pm}v)_{\boldsymbol{k}} = v_{(\tau_i^{\pm}\boldsymbol{k})}, \quad \boldsymbol{k} \in \mathfrak{N} \text{ (resp. } \overline{\mathfrak{N}}^j).$$

A difference operator $\bar{D}_i$ and an averaging operator $\bar{A}_i$ are then given by

$$\begin{aligned}(\bar{D}_i v)_{\boldsymbol{k}} &= (h_{i,k_i})^{-1}((\tau_i^+v)_{\boldsymbol{k}} - (\tau_i^-v)_{\boldsymbol{k}}), \quad \boldsymbol{k} \in \mathfrak{N} \text{ (resp. } \overline{\mathfrak{N}}^j),\\ (\bar{A}_i v)_{\boldsymbol{k}} &= \overline{v}^i_{\boldsymbol{k}} = \frac{1}{2}((\tau_i^+v)_{\boldsymbol{k}} + (\tau_i^-v)_{\boldsymbol{k}}), \quad \boldsymbol{k} \in \mathfrak{N} \text{ (resp. } \overline{\mathfrak{N}}^j).\end{aligned}$$

Both map $\mathbb{C}^{\overline{\mathfrak{M}}^i} \to \mathbb{C}^{\mathfrak{M}}$ and $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}} \to \mathbb{C}^{\overline{\mathfrak{M}}^j}$.

1.1.7. Sampling of continuous functions. A continuous function f defined on $\overline{\Omega}$ can be sampled on the primal mesh $f^{\mathfrak{M}} = \{f(\boldsymbol{x_k});\ \boldsymbol{k} \in \mathfrak{N}\}$, which we identify to

$$f^{\mathfrak{M}} = \sum_{\boldsymbol{k}\in\mathfrak{N}} \mathbf{1}_{b_{\boldsymbol{k}}} f_{\boldsymbol{k}}, \quad f_{\boldsymbol{k}} = f(\boldsymbol{x_k}), \quad \boldsymbol{k} \in \mathfrak{N},$$

with $b_{\boldsymbol{k}}$ as defined in (1.4). We also set

$$f^{\partial\mathfrak{M}} = \{f(\boldsymbol{x_k});\ \boldsymbol{k} \in \partial\mathfrak{N}\}, \quad f^{\mathfrak{M}\cup\partial\mathfrak{M}} = \{f(\boldsymbol{x_k});\ \boldsymbol{k} \in \mathfrak{N}\cup\partial\mathfrak{N}\}.$$

The function f can also be sampled on the dual meshes, e.g. $\overline{\mathfrak{M}}^i$, $f^{\overline{\mathfrak{M}}^i} = \{f(\boldsymbol{x_k});\ \boldsymbol{k} \in \overline{\mathfrak{N}}^i\}$ which we identify to

$$f^{\overline{\mathfrak{M}}^i} = \sum_{\boldsymbol{k}\in\overline{\mathfrak{N}}^i} \mathbf{1}_{\overline{b}^i_{\boldsymbol{k}}} f_{\boldsymbol{k}}, \quad f_{\boldsymbol{k}} = f(\boldsymbol{x_k}), \quad \boldsymbol{k} \in \overline{\mathfrak{N}}^i$$

with similar definitions for $f^{\partial\overline{\mathfrak{M}}^i}$, $f^{\overline{\mathfrak{M}}^i\cup\partial\overline{\mathfrak{M}}^i}$ and sampling on the meshes $\overline{\mathfrak{M}}^{ij}$, $\overline{\mathfrak{M}}^{ij} \cup \partial\overline{\mathfrak{M}}^{ij}$.

In the sequel, we shall use the symbol f for both the continuous function and its sampling on the primal or dual meshes. In fact, from the context, one will be able to deduce the appropriate sampling. For example, with u defined on the primal mesh, $\mathfrak{M}$, in the following expression, $\bar{D}_i(\gamma D_i u)$, it is clear that the function γ is sampled on the dual mesh $\overline{\mathfrak{M}}^i$ as $D_i u$ is defined on this mesh and the operator $\bar{D}_i$ acts on functions defined on this mesh.

To evaluate the action of multiple iterations of discrete operators, e.g. $D_i, \bar{D}_i, A_i, \bar{A}_i$ on a continuous function we may require the function to be defined in a neighborhood of $\overline{\Omega}$. This will be the case here of the diffusion coefficients in the elliptic operator and the Carleman weight function we shall introduce. For a function f defined on a neighborhood of $\overline{\Omega}$ we set

$$\tau_i^{\pm} f(\boldsymbol{x}) := f\Big(\boldsymbol{x} \pm \frac{h_i}{2}\boldsymbol{e}_i\Big), \qquad \boldsymbol{e}_i = (\delta_{i1}, \dots, \delta_{id}),$$
$$\mathsf{D}_i f := (h_i)^{-1}(\tau_i^+ - \tau_i^-) f, \quad \mathsf{A}_i f = \hat{f}^i = \frac{1}{2}(\tau_i^+ + \tau_i^-) f.$$

For a function f continuously defined in a neighborhood of $\overline{\Omega}$, the discrete function $D_i f$ is in fact $\mathsf{D}_i f$ sampled on the dual mesh, $\overline{\mathfrak{M}}^i$, and $\bar{D}_i f$ is $\mathsf{D}_i f$ sampled on the primal mesh, $\mathfrak{M}$. We shall use similar meanings for averaging symbols, $\tilde{f}$, $\overline{f}$, and for more general combinations: for instance, if $i \neq j$, $\widetilde{D_j f}^i$, $\overline{D_i \bar{D}_j f}^i$, $\overline{D_i D_j f}^i$ will be respectively the functions $\widehat{\mathsf{D}_j f}^i$ sampled on $\overline{\mathfrak{M}}^{ij}$, $\widehat{\mathsf{D}_i \mathsf{D}_j f}^i$ sampled on $\mathfrak{M}$, and $\widehat{\mathsf{D}_i \mathsf{D}_j f}^i$ sampled on $\overline{\mathfrak{M}}^j$

1.1.8. Regular families of non-uniform meshes. In this paper, we address non uniform meshes that are obtained as the smooth image of an uniform grid.

More precisely, let $\Omega^\star = (0,1)$ and let $\vartheta_i : \mathbb{R} \mapsto \mathbb{R}$, $i \in [\![1, d]\!]$ be increasing maps such that

$$\vartheta_i \in \mathscr{C}^\infty, \quad \vartheta_i(\overline{\Omega^\star}) = [0, L_i], \quad \inf_{\Omega^\star} \vartheta_i' > 0. \tag{1.6}$$

Let $h_i^\star = \frac{1}{N_i+1}$ and $\mathfrak{M}_0$ be the following uniform primal mesh on $[0,1]^d$

$$\mathfrak{M}_0 = \{\boldsymbol{x}^0_{\boldsymbol{k}} = (x^0_{1,k_1}, \dots, x^0_{d,k_d}) = (k_1 h_1^\star, \dots, k_d h_d^\star), \quad \boldsymbol{k} \in \mathfrak{N}\},$$

and $\overline{\mathfrak{M}_0}^i$, $i \in [\![1,d]\!]$ the associated dual meshes.

We define a non uniform mesh on Ω

$$\mathfrak{M} = \{\boldsymbol{x_k},\ \boldsymbol{k} \in \mathfrak{N}\},$$

with

$$\boldsymbol{x_k} = (\vartheta_1(x^0_{1,k_1}), \dots, \vartheta_d(x^0_{d,k_d})) \tag{1.7}$$

We set $h^\star = \sup_{i\in[\![1,d]\!]} h_i^\star$. Once the functions ϑ_i, $i \in [\![1,d]\!]$, are fixed we assume that for some $C > 0$ we have

$$Ch^\star \le h_i^\star \le h^\star, \quad i \in [\![1,d]\!].$$

For the mesh $\mathfrak{M}$, this in turn implies, for some $C' > 0$, for all $i \in [\![1,d]\!]$,

$$C'h \le h_{i,l} \le h, \quad l \in [\![1,N_i]\!], \qquad C'h \le h_{i,l+\frac{1}{2}} \le h, \quad l \in [\![0,N_i]\!].$$

In particular,

$$C'h \le h_i \le h, \quad i \in [\![1,d]\!]. \tag{1.8}$$

We define the following quantities in order to measure the regularity of the meshes under study

$$\operatorname{reg}(\vartheta_i) = \max\left(\sup_{\Omega^\star} \vartheta_i', \sup_{\Omega^\star} (\vartheta_i')^{-1}, \sup_{\Omega^\star} |\vartheta_i''|\right),$$

$$\operatorname{reg}(\vartheta) = \prod_{i=1}^{d} \operatorname{reg}(\vartheta_i).$$

Note that $\operatorname{reg}(\vartheta_i) \ge 1$ for any $i \in [\![1,d]\!]$.

We shall call uniform meshes, the regular meshes that are obtained with the following linear choice: $\vartheta_i(x) = L_i x$.

1.1.9. Additional notation. We shall denote by z^* the complex conjugate of $z \in \mathbb{C}$. In the sequel, C will denote a generic constant independent of h, whose value may change from line to line. As usual, we shall denote by $\mathcal{O}(1)$ a bounded function. We shall denote by $\mathcal{O}_\mu(1)$ a function that depends on a parameter μ and is *bounded* once μ is fixed. The notation C_μ will denote a constant whose value depends on the parameter μ.

We say that α is a multi-index if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. For $\alpha \in \mathbb{N}^n$ and $\xi \in \mathbb{R}^n$ we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad \partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

1.2. Statement of the main results. With the notation we have introduced, a consistent finite-difference approximation of $\mathcal{A}u$ with homogeneous boundary conditions is

$$\mathcal{A}^{\mathfrak{M}} u = -\sum_{i\in[\![1,d]\!]} \bar{D}_i(\gamma_i D_i u)$$

for $u \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ satisfying $u_{|\partial\Omega} = u^{\partial\mathfrak{M}} = 0$. Recall that, in each term, γ_i is the sampling of the given continuous diffusion coefficient γ_i on the dual mesh $\overline{\mathfrak{M}}^i$, so that for any $u \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ we have

$$(\mathcal{A}^{\mathfrak{M}} u)_{\boldsymbol{k}} = - \sum_{i\in[\![1,d]\!]} \frac{\gamma_i\Big(\boldsymbol{x}_{\tau_i^+\boldsymbol{k}}\Big)\frac{\big((\tau_i^+)^2u\big)_{\boldsymbol{k}}-u_{\boldsymbol{k}}}{h_{i,k_i+\frac12}} - \gamma_i\Big(\boldsymbol{x}_{\tau_i^-\boldsymbol{k}}\Big)\frac{u_{\boldsymbol{k}}-\big((\tau_i^-)^2u\big)_{\boldsymbol{k}}}{h_{i,k_i-\frac12}}}{h_{i,k_i}}, \quad \boldsymbol{k}\in\mathfrak{N}.$$

Note however that other consistent choices of discretization of γ_i on the dual meshes are possible, such as the averaging on the dual mesh $\overline{\mathfrak{M}}^i$ of the sampling of γ_i on the primal mesh. Our results also holds for such discrete operators.

REMARK 1.2. *Finite differences are not well adapted to address anisotropic elliptic operators. Here, we only treat the case of a diagonal anisotropic operator, i.e. an anisotropy associated with the principal axes. Note however that the treatment we make of non uniform meshes naturally leads to such diagonal anisotropic operators by a change of variables, even starting from an isotropic diffusion coefficient.*

We choose a function ψ that satisfies the following properties.

ASSUMPTION 1.3. *Let $\tilde\Omega$ be a smooth open and connected bounded neighborhood of $\overline{\Omega}$ in $\mathbb{R}^d$ and set $\tilde Q = (0,\mathcal{T})\times\tilde\Omega$. The function ψ is in $\mathscr{C}^p(\overline{\tilde Q},\mathbb{R})$, with p sufficiently large, and satisfies, for some $c>0$,*

$$|\nabla\psi| \geq c \text{ and } \psi > 0 \text{ in } \tilde Q,$$
$$\partial_{n_i}\psi(t,\boldsymbol{x}) < 0 \text{ in } (0,\mathcal{T})\times V_{\partial_i\Omega}, \quad \partial_i^2\psi(t,\boldsymbol{x}) \geq 0 \text{ in } (0,\mathcal{T})\times V_{\partial_i\Omega},$$
$$\partial_t\psi \geq c \text{ on } \{0\}\times(\Omega\setminus\omega), \quad \psi = \mathrm{Cst} \text{ and } \partial_t\psi \leq -c \text{ on } \{\mathcal{T}\}\times\Omega,$$

where $V_{\partial_i\Omega}$ is a sufficiently small neighborhood of $\partial_i\Omega$ in $\tilde\Omega$, in which the outward unit normal n_i to Ω is extended from $\partial_i\Omega$. The construction of such a weight function is described in Section A. We then set $\varphi = e^{\lambda\psi}$.

To state the Carleman estimate for the semi-discrete operator $-\partial_t^2 + \mathcal{A}^{\mathfrak{M}}$, we introduce the following discrete gradient operator $\Upsilon = (D_1,\dots,D_d)^t$.

THEOREM 1.4. *Let ϑ_i, $i\in[\![1,d]\!]$ satisfy (1.6) and ψ be a weight function satisfying (1.3) for the observation domain ω. For the parameter $\lambda\geq 1$ sufficiently large, there exist C, $s_0\geq 1$, $h_0>0$, $\varepsilon_0>0$, depending on ω, $\mathcal{T}$, $(\vartheta_i)_{i\in[\![1,d]\!]}$ and $\mathrm{reg}(\Gamma)$, such that for any mesh $\mathfrak{M}$ obtained from $(\vartheta_i)_{i\in[\![1,d]\!]}$ by (1.7), we have*

$$\begin{aligned} s^3\|e^{s\varphi}u\|^2_{L^2(Q)} + s\|e^{s\varphi}\partial_t u\|^2_{L^2(Q)} + s\|e^{s\varphi}\Upsilon u\|^2_{L^2(Q)} + s|e^{s\varphi(0,.)}\partial_t u(0,.)|^2_{L^2(\Omega)} \\ + se^{2s\varphi(\mathcal{T})}|\partial_t u(\mathcal{T},.)|^2_{L^2(\Omega)} + s^3e^{2s\varphi(\mathcal{T})}|u(\mathcal{T},.)|^2_{L^2(\Omega)} \\ \leq C\Big(\|e^{s\varphi}(-\partial_t^2+\mathcal{A}^{\mathfrak{M}})u\|^2_{L^2(Q)} + se^{2s\varphi(\mathcal{T})}|\Upsilon u(\mathcal{T},.)|^2_{L^2(\Omega)} \\ + s|e^{s\varphi(0,.)}\partial_t u(0,.)|^2_{L^2(\omega)}\Big), \end{aligned} \tag{1.9}$$

for all $s\geq s_0$, $0<h\leq h_0$ and $sh\leq\varepsilon_0$, and $u\in\mathscr{C}^2([0,\mathcal{T}],\mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}})$, satisfying $u_{|\{0\}\times\Omega}=0$, $u_{|(0,\mathcal{T})\times\partial\Omega}=0$.

Denoting by $\phi^{\mathfrak{M}}$ a set of discrete L^2 orthonormal eigenfunctions, $\phi_j\in\mathbb{C}^{\mathfrak{M}}$, $1\leq j\leq|\mathfrak{M}|$, of the operator $\mathcal{A}^{\mathfrak{M}}$ with homogeneous Dirichlet boundary conditions, and by $\mu^{\mathfrak{M}}$ the set of the associated eigenvalues sorted in a non-decreasing sequence, μ_j, $1\leq j\leq|\mathfrak{M}|$ we have the following result.

THEOREM 1.5 (Partial discrete Lebeau-Robbiano inequality). *Let ϑ satisfying* (1.6)*. There exist $C > 0$, $\varepsilon_1 > 0$ and h_0 such that, for any mesh $\mathfrak{M}$ obtained from ϑ by* (1.7) *such that $h \leq h_0$, for all $0 < \mu \leq \varepsilon_1/h^2$, we have*

$$\sum_{\substack{\mu_k \in \mu^{\mathfrak{M}} \\ \mu_k \leq \mu}} |\alpha_k|^2 = \int_\Omega \Big| \sum_{\substack{\mu_k \in \mu^{\mathfrak{M}} \\ \mu_k \leq \mu}} \alpha_k \phi_k \Big|^2 \leq C e^{C\sqrt{\mu}} \int_\omega \Big| \sum_{\substack{\mu_k \in \mu^{\mathfrak{M}} \\ \mu_k \leq \mu}} \alpha_k \phi_k \Big|^2, \qquad \forall (\alpha_k)_{1 \leq k \leq |\mathfrak{M}|} \subset \mathbb{C}.$$

The proof is given in [BHL09a, Section 6] following the approach introduced in [Le 07].

We introduce the following finite dimensional spaces

$$E_j = \operatorname{Span}\{\phi_k;\ 1 \leq \mu_k \leq 2^{2j}\} \subset \mathbb{C}^{\mathfrak{M}}, \quad j \in \mathbb{N},$$

and denote by Π_{E_j} the L^2-orthogonal projection onto E_j. The controllability result we can deduce from the above results is the following.

THEOREM 1.6. *Let $T > 0$ and ϑ satisfying* (1.6)*. There exist $h_0 > 0$, $C_T > 0$ and C_1, C_2, $C_3 > 0$ such that for all meshes $\mathfrak{M}$ defined by* (1.7)*, with $0 < h \leq h_0$, and all initial data $y_0 \in \mathbb{C}^{\mathfrak{M}}$, there exists a semi-discrete control function v such that the solution to*

$$\partial_t y - \sum_{i \in [\![1,d]\!]} \bar{D}_i(\gamma_i D_i y) = \mathbf{1}_\omega v, \quad y^{\partial\mathfrak{M}} = 0, \quad y|_{t=0} = y_0. \tag{1.10}$$

satisfies $\Pi_{E_{j^{\mathfrak{M}}}} y(T) = 0$, for $j^{\mathfrak{M}} = \max\{j;\ 2^{2j} \leq C_1/h^2\}$, with $\|v\|_{L^2(Q)} \leq C_T |y_0|_{L^2(\Omega)}$ and furthermore $|y(T)|_{L^2(\Omega)} \leq C_2 e^{-C_3/h^2} |y_0|_{L^2(\Omega)}$.

For a proof see [BHL09a, Section 7].

Finally, in the spirit of the work of [LT06] the controllability result we have obtained yields the following relaxed observability estimate

COROLLARY 1.7. *There exist $C_T > 0$ and $C > 0$ depending on Ω, ω, T, and ϑ, such that the semi-discrete solution q in $\mathscr{C}^\infty([0,T], \mathbb{C}^{\mathfrak{M}})$ to*

$$\begin{cases} -\partial_t q + \mathcal{A}^{\mathfrak{M}} q = 0 & \text{in } (0,T) \times \Omega, \\ q = 0 & \text{on } (0,T) \times \partial\Omega, \\ q(T) = q_F \in \mathbb{C}^{\mathfrak{M}}, \end{cases}$$

in the case $h \leq h_0$, satisfies

$$|q(0)|_{L^2(\Omega)} \leq C_T \Big(\int_0^T \int_\omega |q(t)|^2\ dt \Big)^{\frac{1}{2}} + C e^{-C/h^2} |q_F|_{L^2(\Omega)}.$$

As mentioned above, these results can also be used for the analysis of the space/time discretized parabolic control problem [BHL09b].

1.3. Outline. In Section 2 we have gathered preliminary discrete calculus results. Many of the proofs of these results can be found in [BHL09a]. Additional proofs have been placed in Appendix B to ease the reading. Section 3 is devoted to the proof of the semi-discrete elliptic Carleman estimate for uniform meshes. Again, to ease the reading, a large number of proofs of intermediate estimates have been placed in Appendix C. This result is then extended to non-uniform meshes in Section 4. For completeness, in Appendix D we give the counterpart of the Carleman estimate of Theorem 1.4 in the case of a fully-discrete elliptic operator. This result will be used in [BHL10] for the treatment of semi-discrete parabolic operators.

2. Some preliminary discrete calculus results. Here, to prepare for Section 3, we only consider uniform meshes, *i.e.*, constant-step discretizations in each direction, *i.e.*, $h_{i,j+\frac{1}{2}} = h_i = \frac{L_i}{N_i+1}$, $j \in [\![0, N_i]\!]$, $i \in [\![1, d]\!]$.

This section aims to provide calculus rules for discrete operators such as D_i, $\bar{D}_i$ and also to provide estimates for the successive applications of such operators on the weight functions.

2.1. Discrete calculus formulae. We present calculus results for the finite-difference operators that were defined in the introductory section. Proofs are similar to that given in the one-dimension case in [BHL09a].

LEMMA 2.1. *Let the functions f_1 and f_2 be continuously defined in a neighborhood of $\overline{\Omega}$. For $i \in [\![1, d]\!]$, we have*

$$\mathsf{D}_i(f_1 f_2) = \mathsf{D}_i(f_1)\, \hat{f}_2^{\,i} + \hat{f}_1^{\,i}\, \mathsf{D}_i(f_2).$$

Note that the immediate translation of the proposition to discrete functions $f_1, f_2 \in \mathbb{C}^{\mathfrak{M}}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^j}$, $j \neq i$), and $g_1, g_2 \in \mathbb{C}^{\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}}$, $j \neq i$) is

$$D_i(f_1 f_2) = D_i(f_1)\, \tilde{f}_2^{\,i} + \tilde{f}_1^{\,i}\, D_i(f_2), \qquad \bar{D}_i(g_1 g_2) = \bar{D}_i(g_1)\, \overline{g}_2^{\,i} + \overline{g}_1^{\,i}\, \bar{D}_i(g_2).$$

LEMMA 2.2. *Let the functions f_1 and f_2 be continuously defined in a neighborhood of $\overline{\Omega}$. For $i \in [\![1, d]\!]$, we have*

$$\widehat{f_1 f_2}^{\,i} = \hat{f}_1^{\,i} \hat{f}_2^{\,i} + \frac{h_i^2}{4} \mathsf{D}_i(f_1) \mathsf{D}_i(f_2).$$

Note that the immediate translation of the proposition to discrete functions $f_1, f_2 \in \mathbb{C}^{\mathfrak{M}}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^j}$, $j \neq i$), and $g_1, g_2 \in \mathbb{C}^{\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}^{ij}}$, $j \neq i$)

$$\widetilde{f_1 f_2}^{\,i} = \tilde{f}_1^{\,i} \tilde{f}_2^{\,i} + \frac{h_i^2}{4} D_i(f_1) D_i(f_2), \qquad \overline{g_1 g_2}^{\,i} = \overline{g}_1^{\,i} \overline{g}_2^{\,i} + \frac{h_i^2}{4} \bar{D}_i(g_1) \bar{D}_i(g_2).$$

Some of the following properties can be extended in such a manner to discrete functions. We shall not always write it explicitly.

Averaging a function twice gives the following formula.

LEMMA 2.3. *Let the function f be continuously defined in a neighborhood of $\overline{\Omega}$. For $i \in [\![1, d]\!]$ we have*

$$\mathsf{A}_i^2 f := \hat{\hat{f}}^{\,i\,i} = f + \frac{h_i^2}{4} \mathsf{D}_i \mathsf{D}_i f.$$

The following proposition covers discrete integrations by parts and related formulae.

PROPOSITION 2.4. *Let $f \in \mathbb{C}^{\mathfrak{M} \cup \partial \mathfrak{M}}$ and $g \in \mathbb{C}^{\overline{\mathfrak{M}}^i}$. For $i \in [\![1, d]\!]$ we have*

$$\iint_\Omega f (\bar{D}_i g) = - \iint_\Omega (D_i f) g + \int_{\Omega_i} (f_{N_i+1} g_{N_i+\frac{1}{2}} - f_0 g_{\frac{1}{2}}),$$

$$\iint_\Omega f \overline{g}^{\,i} = \iint_\Omega \tilde{f}^{\,i} g - \frac{h_i}{2} \int_{\Omega_i} (f_{N_i+1} g_{N_i+\frac{1}{2}} + f_0 g_{\frac{1}{2}}).$$

LEMMA 2.5. *Let* $i \in [\![1, d]\!]$ *and* $v \in \mathbb{C}^{\mathfrak{M} \cup \partial \mathfrak{M}}$ *(resp.* $\mathbb{C}^{\overline{\mathfrak{M}}^j \cup \partial \overline{\mathfrak{M}}^j}$ *for* $j \neq i$*) be such that* $v_{|\partial_i \Omega} = 0$. *Then* $\iint_\Omega v = \iint_\Omega \tilde{v}^i$.

LEMMA 2.6. *Let* f *be a smooth function defined in a neighborhood of* $\overline{\Omega}$. *For* $i \in [\![1, d]\!]$ *we have*

$$\tau_i^{\pm} f = f \pm \frac{h_i}{2} \int_0^1 \partial_i f(. \pm \sigma \boldsymbol{h}_i/2)\, d\sigma, \quad \mathsf{A}_i^\ell f = f + C_\ell h_i^2 \int_{-1}^1 (1 - |\sigma|)\, \partial_i^2 f(. + l_\ell \sigma \boldsymbol{h}_i)\, d\sigma,$$

$$\mathsf{D}_i^\ell f = \partial_i^\ell f + C'_\ell h_i^2 \int_{-1}^1 (1 - |\sigma|)^{\ell+1}\, \partial_i^{\ell+2} f(. + l_\ell \sigma \boldsymbol{h}_i)\, d\sigma, \quad \ell = 1, 2, \quad l_1 = \frac{1}{2},\ l_2 = 1,$$

with $\boldsymbol{h}_i = h_i \boldsymbol{e}_i$.

For $i, j \in [\![1, d]\!]$, $i \neq j$, *we have*

$$\begin{aligned} \mathsf{D}_i \mathsf{D}_j f = \partial_{ij}^2 f &+ C'' \frac{|\boldsymbol{h}_{ij}^+|^4}{h_i h_j} \int_{-1}^1 (1 - |\sigma|)^3 f^{(4)}(. + \sigma \boldsymbol{h}_{ij}^+/2; \boldsymbol{\eta}^+, \ldots, \boldsymbol{\eta}^+)\, d\sigma \\ &+ C''' \frac{|\boldsymbol{h}_{ij}^+|^4}{h_i h_j} \int_{-1}^1 (1 - |\sigma|)^3 f^{(4)}(x + \sigma \boldsymbol{h}_{ij}^-/2; \boldsymbol{\eta}^-, \ldots, \boldsymbol{\eta}^-)\, d\sigma, \end{aligned}$$

with $\boldsymbol{h}_{ij}^{\pm} = h_i \boldsymbol{e}_i \pm h_j \boldsymbol{e}_j$ *and* $\boldsymbol{\eta}^{\pm} = \frac{1}{|\boldsymbol{h}_{ij}^{\pm}|}(\boldsymbol{h}_{ij}^{\pm})$.

Note that $\frac{|\boldsymbol{h}_{ij}^+|^4}{h_i h_j} = \mathcal{O}(h^2)$ by (1.8), for $i, j \in [\![1, d]\!]$, $j \neq i$.

Proof. This series of results follow from Taylor formulae,

$$f(\boldsymbol{x} + \boldsymbol{\eta}) = \sum_{j=0}^{n-1} \frac{1}{j!} f^{(j)}(\boldsymbol{x}; \boldsymbol{\eta}, \ldots, \boldsymbol{\eta}) + \int_0^1 \frac{(1 - \sigma)^{n-1}}{(n-1)!} f^{(n)}(\boldsymbol{x} + \sigma \boldsymbol{\eta}; \boldsymbol{\eta}, \ldots, \boldsymbol{\eta})\, d\sigma,$$

at order $n = 1$, $n = 2$, $n = 3$ or $n = 4$. ∎

2.2. Calculus results related to the weight functions. We now present some technical lemmata related to discrete operations performed on the Carleman weight function that is of the form $e^{s\varphi}$ with $\varphi = e^{\lambda \psi}$, $\psi \in \mathscr{C}^p$, with p sufficiently large. For concision, we set $r = e^{s\varphi}$ and $\rho = r^{-1}$. The positive parameters s and h will be large and small respectively and we are particularly interested in the dependence on s, h and λ in the following basic estimates.

We assume $s \geq 1$ and $\lambda \geq 1$. We shall use multi-indices of the form $\alpha = (\alpha_t, \alpha_x)$ with $\alpha_t \in \mathbb{N}$ and $\alpha_x \in \mathbb{N}^d$.

LEMMA 2.7. *Let* α *and* β *be multi-indices. We have*

$$\begin{aligned} \partial^\beta (r \partial^\alpha \rho) = &|\alpha|^{|\beta|} (-s\varphi)^{|\alpha|} \lambda^{|\alpha + \beta|} (\nabla \psi)^{\alpha + \beta} \\ &+ |\alpha||\beta| (s\varphi)^{|\alpha|} \lambda^{|\alpha+\beta|-1} \mathcal{O}(1) + s^{|\alpha|-1} |\alpha|(|\alpha| - 1) \mathcal{O}_\lambda(1) = \mathcal{O}_\lambda(s^{|\alpha|}). \end{aligned} \tag{2.1}$$

Let $\sigma \in [-1, 1]$ *and* $i \in [\![1, d]\!]$. *We have*

$$\partial^\beta (r(\boldsymbol{x})(\partial^\alpha \rho)(\boldsymbol{x} + \sigma \boldsymbol{h}_i)) = \mathcal{O}_\lambda(s^{|\alpha|}(1 + (sh)^{|\beta|}))\ e^{\mathcal{O}_\lambda(sh)}. \tag{2.2}$$

Provided $sh \leq \mathfrak{K}$ *we have* $\partial^\beta (r(\boldsymbol{x})(\partial^\alpha \rho)(\boldsymbol{x} + \sigma \boldsymbol{h}_i)) = \mathcal{O}_{\lambda, \mathfrak{K}}(s^{|\alpha|})$. *The same expressions hold with* r *and* ρ *interchanged and with* s *changed into* $-s$.

For a proof see [BHL09a, proof of Lemma 3.7].

With Leibniz formula we have the following estimate.

COROLLARY 2.8. *Let α, β and δ be multi-indices. We have*

$$\begin{aligned}\partial^\delta(r^2(\partial^\alpha\rho)\partial^\beta\rho) =& |\alpha+\beta|^{|\delta|}(-s\varphi)^{|\alpha+\beta|}\lambda^{|\alpha+\beta+\delta|}(\nabla\psi)^{\alpha+\beta+\delta}\\ &+|\delta||\alpha+\beta|(s\varphi)^{|\alpha+\beta|}\lambda^{|\alpha+\beta+\delta|-1}\mathcal{O}(1)\\ &+s^{|\alpha+\beta|-1}(|\alpha|(|\alpha|-1)+|\beta|(|\beta|-1))\mathcal{O}_\lambda(1)=\mathcal{O}_\lambda(s^{|\alpha+\beta|}).\end{aligned}$$

The proofs of the following properties can be found in Appendix B.

PROPOSITION 2.9. *Let α be a multi-index. Let $i,j\in[\![1,d]\!]$, provided $sh\le\mathfrak{K}$, we have*

$$\begin{aligned} r\tau_i^{\pm}\partial^\alpha\rho &= r\partial^\alpha\rho+s^{|\alpha|}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)=s^{|\alpha|}\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\ r\mathsf{A}_i^k\partial^\alpha\rho &= r\partial^\alpha\rho+s^{|\alpha|}\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)=s^{|\alpha|}\mathcal{O}_{\lambda,\mathfrak{K}}(1),\quad k=1,2,\\ r\mathsf{A}_i^k\mathsf{D}_i\rho &= r\partial_x\rho+s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)=s\mathcal{O}_{\lambda,\mathfrak{K}}(1),\quad k=0,1,\\ r\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\rho &= r\partial_i^{k_i}\partial_j^{k_j}\rho+s^2\,\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)=s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1),\quad k_i+k_j\le 2.\end{aligned}$$

The same estimates hold with ρ and r interchanged.

LEMMA 2.10. *Let α and β be multi-indices and $k\in\mathbb{N}$. Let $i,j\in[\![1,d]\!]$, provided $sh\le\mathfrak{K}$, we have*

$$\begin{aligned}&\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}(\partial^\beta(r\partial^\alpha\rho))=\partial_i^{k_i}\partial_j^{k_j}\partial^\beta(r\partial^\alpha\rho)+h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|}),\quad k_i+k_j\le 2,\\ &\mathsf{A}_i^k\partial^\beta(r\partial^\alpha\rho)=\partial^\beta(r\partial^\alpha\rho)+h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|}).\end{aligned}$$

Let $\sigma\in[-1,1]$, we have $\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\beta(r(x)\partial^\alpha\rho(x+\sigma h_i))=\mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|})$, for $k_i+k_j\le 2$. The same estimates hold with r and ρ interchanged.

LEMMA 2.11. *Let α, β and δ be multi-indices and $k\in\mathbb{N}$. Let $i,j\in[\![1,d]\!]$, provided $sh\le\mathfrak{K}$, we have*

$$\begin{aligned}\mathsf{A}_i^k\partial^\delta(r^2(\partial^\alpha\rho)\partial^\beta\rho)&=\partial^\delta(r^2(\partial^\alpha\rho)\partial^\beta\rho)+h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|+|\beta|})=\mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|+|\beta|}),\\ \mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\delta(r^2(\partial^\alpha\rho)\partial^\beta\rho)&=\partial_i^{k_i}\partial_j^{k_j}(\partial^\delta(r^2(\partial^\alpha\rho)\partial^\beta\rho))+h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|+|\beta|})\\ &=\mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|+|\beta|}),\quad k_i+k_j\le 2.\end{aligned}$$

Let $\sigma,\sigma'\in[-1,1]$. We have

$$\begin{aligned}&\mathsf{A}_i^k\partial^\delta\left(r(\boldsymbol{x})^2(\partial^\alpha\rho(\boldsymbol{x}+\sigma\boldsymbol{h}_i))\partial^\beta\rho(\boldsymbol{x}+\sigma'\boldsymbol{h}_j)\right)=\mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|+|\beta|}),\\ &\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\delta\left(r(\boldsymbol{x})^2(\partial^\alpha\rho(\boldsymbol{x}+\sigma\boldsymbol{h}_i))\partial^\beta\rho(\boldsymbol{x}+\sigma'\boldsymbol{h}_j)\right)=\mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|+|\beta|}),\quad k_i+k_j\le 2.\end{aligned}$$

The same estimates hold with r and ρ interchanged.

PROPOSITION 2.12. *Let α be a multi-index and $k\in\mathbb{N}$. Let $i,j\in[\![1,d]\!]$, provided $sh\le\mathfrak{K}$, we have*

$$\begin{aligned}&\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\mathsf{A}_i^k\partial^\alpha(r\widehat{\mathsf{D}_i\rho}^{\,i})=\partial_i^{k_i}\partial_j^{k_j}\partial^\alpha(r\partial_x\rho)+s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)=s\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\ &\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}(r\mathsf{D}_i^2\rho)=\partial_i^{k_i}\partial_j^{k_j}(r\partial_i^2\rho)+s^2\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)=s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\ &\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}(r\mathsf{A}_i^2\rho)=\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2).\end{aligned}$$

The same estimates hold with r and ρ interchanged.

PROPOSITION 2.13. *Let α, β be multi-indices, $i, j \in [\![1, d]\!]$ and $k_i, k_i', k_j, k_j' \in \mathbb{N}$. For $k_i + k_j \leq 2$, provided $sh \leq \mathfrak{K}$ we have*

$$\begin{aligned}
\mathsf{A}_i^{k_i'} \mathsf{A}_j^{k_j'} \mathsf{D}_i^{k_i} \mathsf{D}_j^{k_j} \partial^\beta (r^2 (\partial^\alpha \rho) \widehat{\mathsf{D}_i \rho}^{\,i}) &= \partial_i^{k_i} \partial_j^{k_j} \partial^\beta (r^2 (\partial^\alpha \rho) \partial_i \rho) + s^{|\alpha|+1} \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) \\
&= s^{|\alpha|+1} \mathcal{O}_{\lambda,\mathfrak{K}}(1), \\
\mathsf{A}_i^{k_i'} \mathsf{A}_j^{k_j'} \mathsf{D}_i^{k_i} \mathsf{D}_j^{k_j} \partial^\beta (r^2 (\partial^\alpha \rho) \mathsf{A}_i^2 \rho) &= \partial_i^{k_i} \partial_j^{k_j} \partial^\beta (r (\partial^\alpha \rho)) + s^{|\alpha|} \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) \\
&= s^{|\alpha|} \mathcal{O}_{\lambda,\mathfrak{K}}(1), \\
\mathsf{A}_i^{k_i'} \mathsf{A}_j^{k_j'} \mathsf{D}_i^{k_i} \mathsf{D}_j^{k_j} \partial^\beta (r^2 (\partial^\alpha \rho) \mathsf{D}_i^2 \rho) &= \partial_i^{k_i} \partial_j^{k_j} \partial^\beta (r^2 (\partial^\alpha \rho) \partial_i^2 \rho) + s^{|\alpha|+2} \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) \\
&= s^{|\alpha|+2} \mathcal{O}_{\lambda,\mathfrak{K}}(1),
\end{aligned}$$

and we have

$$\begin{aligned}
&\mathsf{A}_i^{k_i'} \mathsf{A}_j^{k_j'} \mathsf{D}_i^{k_i} \mathsf{D}_j^{k_j} \partial^\alpha (r^2 \widehat{\mathsf{D}_i \rho}^{\,i} \mathsf{D}_j^2 \rho) = \partial_i^{k_i} \partial_j^{k_j} \partial^\alpha (r^2 (\partial_i \rho) \partial_j^2 \rho) + s^3 \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) = s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(1), \\
&\mathsf{A}_i^{k_i'} \mathsf{A}_j^{k_j'} \mathsf{D}_i^{k_i} \mathsf{D}_j^{k_j} \partial^\alpha (r^2 \widehat{\mathsf{D}_i \rho}^{\,i} \mathsf{A}_j^2 \rho) = \partial_i^{k_i} \partial_j^{k_j} \partial^\alpha (r \partial_i \rho) + s \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) = s \mathcal{O}_{\lambda,\mathfrak{K}}(1).
\end{aligned}$$

3. A semi-discrete elliptic Carleman estimate for uniform meshes.

3. A semi-discrete elliptic Carleman estimate for uniform meshes. Here we consider constant-step discretizations in each direction. The case of regular non-uniform meshes is treated in Section 4.

In preparation to this section, we shall prove here the Carleman estimate on uniform meshes, for a slightly more general semi-discrete elliptic operator that we define now. For all $i \in [\![1, d]\!]$, let $\xi_{1,i} \in \mathbb{R}^{\mathfrak{M}}$ and $\xi_{2,i} \in \mathbb{R}^{\overline{\mathfrak{M}}^i}$ be two positive discrete functions. We denote by $\operatorname{reg}(\xi)$ the following quantity

$$\operatorname{reg}(\xi) = \max_{i \in [\![1,d]\!]} \operatorname{reg}(\xi_{1,i}, \xi_{2,i}), \tag{3.1}$$

with

$$\begin{aligned}
\operatorname{reg}(\xi_{1,i}, \xi_{2,i}) = \max \Bigg(&\sup_{\mathfrak{M}} \Big(\xi_{1,i} + \frac{1}{\xi_{1,i}} \Big), \ \sup_{\overline{\mathfrak{M}}^i} \Big(\xi_{2,i} + \frac{1}{\xi_{2,i}} \Big), \\
&\max_{j \in [\![1,d]\!]} \sup_{\overline{\mathfrak{M}}^j} |D_j \xi_{1,i}|, \ \sup_{\mathfrak{M}} |\bar{D}_i \xi_{2,i}|, \ \max_{\substack{j \in [\![1,d]\!] \\ i \neq j}} \sup_{\overline{\mathfrak{M}}^{ij}} |D_j \xi_{2,i}| \Bigg).
\end{aligned} \tag{3.2}$$

Hence, $\operatorname{reg}(\xi)$ measures the boundedness of $\xi_{1,i}$ and $\xi_{2,i}$ and of their discrete derivatives as well as the distance to zero of $\xi_{1,i}$ and $\xi_{2,i}$, $i \in [\![1, d]\!]$.

By abuse of notation, the letters $\xi_{1,i}, \xi_{2,i}$ will also refer to a $\mathbb{Q}^1$-interpolation of these values on $\mathfrak{M}$ and $\overline{\mathfrak{M}}^i$ respectively. Note that the resulting interpolated functions are Lipschitz continuous with

$$\|\xi_{1,i}\|_{W^{1,\infty}} \leq C \operatorname{reg}(\xi), \quad \|\xi_{2,i}\|_{W^{1,\infty}} \leq C \operatorname{reg}(\xi).$$

We introduce the following notation related to the coefficients $\xi_{1,i}$ and $\xi_{2,i}$, for any function f

$$\begin{aligned} D_{i,\xi} f &= \sqrt{\xi_{1,i}\xi_{2,i}} D_i f, \ i \in [\![1,d]\!] \\ \Upsilon_\xi f &= \left(\sqrt{\xi_{1,1}\xi_{2,1}} D_1 f, \dots, \sqrt{\xi_{1,d}\xi_{2,d}} D_d f\right)^t = \left(D_{1,\xi} f, \dots, D_{d,\xi} f\right)^t, \\ \nabla_\xi f &= \left(\partial_t f, \sqrt{\xi_{1,1}\xi_{2,1}}\partial_{x_1} f, \dots, \sqrt{\xi_{1,d}\xi_{2,d}}\partial_{x_d} f\right)^t = \begin{pmatrix} \partial_t f \\ \Upsilon_\xi f \end{pmatrix}, \\ \Delta_\xi f &= \partial_t^2 f + \sum_{i\in[\![1,d]\!]} \xi_{1,i}\xi_{2,i}\partial_{x_i}^2 f. \end{aligned}$$

We let $\omega \Subset \Omega$ be a nonempty open subset. We set the operator $P^{\mathfrak{M}}$ to be

$$P^{\mathfrak{M}} = -\partial_t^2 - \sum_{i\in[\![1,d]\!]} \xi_{1,i}\bar{D}_i(\xi_{2,i}D_i),$$

continuous in the variable $t \in (0, \mathcal{T})$, with $\mathcal{T} > 0$, and discrete in the variable $\boldsymbol{x} \in \Omega$.

The Carleman weight function is of the form $r = e^{s\varphi}$ with $\varphi = e^{\lambda\psi}$, where ψ satisfies Assumption 1.3.

The enlarged neighborhood $\tilde{\Omega}$ of Ω introduced in Assumption 1.3 allows us to apply multiple discrete operators such as D_i and A_i on the weight functions. In particular, this then yields on $\partial_i\Omega$

$$(r\overline{D_i\rho}^i)_{|k_i=0} \leq 0, \quad (r\overline{D_i\rho}^i)_{|k_i=N_i+1} \geq 0, \quad i \in [\![1,d]\!]. \tag{3.3}$$

We are now in position to state and prove the following semi-discrete Carleman estimate.

THEOREM 3.1. *Let* $\mathrm{reg}^0 > 0$ *be given. For the parameter* $\lambda \geq 1$ *sufficiently large, there exist* C, $s_0 \geq 1$, $h_0 > 0$, $\varepsilon_0 > 0$, *depending on* ω, $\mathcal{T}$, reg^0, *such that for any* $\xi_{1,i}$, $\xi_{2,i}$, $i \in [\![1,d]\!]$, *with* $\mathrm{reg}(\xi) \leq \mathrm{reg}^0$ *we have*

$$\begin{aligned} s^3\|e^{s\varphi}u\|^2_{L^2(Q)} + s\|e^{s\varphi}\partial_t u\|^2_{L^2(Q)} + s\sum_{i\in[\![1,d]\!]}\|e^{s\varphi}D_i u\|^2_{L^2(Q)} + s|e^{s\varphi(0,.)}\partial_t u(0,.)|^2_{L^2(\Omega)} \\ + se^{2s\varphi(\mathcal{T})}|\partial_t u(\mathcal{T},.)|^2_{L^2(\Omega)} + s^3 e^{2s\varphi(\mathcal{T})}|u(\mathcal{T},.)|^2_{L^2(\Omega)} \\ \leq C\left(\|e^{s\varphi}P^{\mathfrak{M}}u\|^2_{L^2(Q)} + s\sum_{i\in[\![1,d]\!]} e^{2s\varphi(\mathcal{T})}|D_i u(\mathcal{T},.)|^2_{L^2(\Omega)} + s|e^{s\varphi(0,.)}\partial_t u(0,.)|^2_{L^2(\omega)}\right), \end{aligned} \tag{3.4}$$

for all $s \geq s_0$, $0 < h \leq h_0$ *and* $sh \leq \varepsilon_0$, *and* $u \in \mathscr{C}^2([0,T], \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}})$, *satisfying* $u_{|\{0\}\times\Omega} = 0$, $u_{|(0,\mathcal{T})\times\partial\Omega} = 0$.

Proof. We set $f := -P^{\mathfrak{M}}u$. At first, we shall work with the function $v = ru$, *i.e.*, $u = \rho v$, that satisfies

$$r\left(\partial_t^2(\rho v) + \sum_{i\in[\![1,d]\!]}\xi_{1,i}\bar{D}_i\big(\xi_{2,i}D_i(\rho v)\big)\right) = rf. \tag{3.5}$$

We have $\partial_t^2(\rho v) = (\partial_t^2\rho)v + 2(\partial_t\rho)\partial_t v + \rho\partial_t^2 v$ and by Lemma 2.1

$$\bar{D}_i(\xi_{2,i}D_i(\rho v)) = (\bar{D}_i(\xi_{2,i}D_i\rho))\,\tilde{\bar{v}}^i + \overline{\xi_{2,i}D_i\rho}^i\,\overline{D_i v}^i + (\overline{D_i\rho}^i)\,\overline{\xi_{2,i}D_i v}^i + \tilde{\bar{\rho}}^i\,\bar{D}_i(\xi_{2,i}D_i v).$$

By Lemma 2.2 we have, for $i \in [\![1, d]\!]$,

$$\begin{aligned}
\overline{\xi_{2,i} D_i v}^i &= \overline{\xi_{2,i}}^i \; \overline{D_i v}^i + \frac{h_i}{4} (\bar{D}_i \xi_{2,i})(\tau_i^+ D_i v - \tau_i^- D_i v),\\
\overline{\xi_{2,i} D_i \rho}^i &= \overline{\xi_{2,i}}^i \; \overline{D_i \rho}^i + \frac{h_i^2}{4} (\bar{D}_i \xi_{2,i})(\bar{D}_i D_i \rho),\\
\bar{D}_i(\xi_{2,i} D_i \rho) &= (\bar{D}_i \xi_{2,i}) \overline{D_i \rho}^i + \overline{\xi_{2,i}}^i \bar{D}_i D_i \rho.
\end{aligned}$$

Using that $\rho r = 1$ and the above equalities, Equation (3.5) thus reads $Av + B_1 v = g'$ with $Av = A_1 v + A_2 v$ where

$$\begin{aligned}
A_1 v &= \partial_t^2 v + \sum_{i \in [\![1,d]\!]} \xi_{1,i} r \overline{\tilde{\rho}}^i \, \bar{D}_i(\xi_{2,i} D_i v),\\
A_2 v &= r(\partial_t^2 \rho)\, v + \sum_{i \in [\![1,d]\!]} \xi_{1,i} \xi_{2,i} r (\bar{D}_i D_i \rho)\, \overline{\tilde{v}}^i,\\
B_1 v &= 2r(\partial_t \rho)\partial_t v + 2 \sum_{i \in [\![1,d]\!]} \xi_{1,i} \xi_{2,i} r \overline{D_i \rho}^i \; \overline{D_i v}^i,\\
g' &= rf - \sum_{i \in [\![1,d]\!]} \frac{h_i}{4} \xi_{1,i} r \overline{D_i \rho}^i (\bar{D}_i \xi_{2,i})(\tau_i^+ D_i v - \tau_i^- D_i v)\\
&\quad - \sum_{i \in [\![1,d]\!]} \frac{h_i^2}{4} \xi_{1,i} (\bar{D}_i \xi_{2,i}) r (\bar{D}_i D_i \rho) \overline{D_i v}^i - h_i \sum_{i \in [\![1,d]\!]} \mathcal{O}(1) r \overline{D_i \rho}^i \; \overline{D_i v}^i\\
&\quad - \sum_{i \in [\![1,d]\!]} \xi_{1,i} \Big(r (\bar{D}_i \xi_{2,i}) \overline{D_i \rho}^i + h_i \mathcal{O}(1) r (\bar{D}_i D_i \rho) \Big) \overline{\tilde{v}}^i,
\end{aligned}$$

since $\|\overline{\xi_{2,i}}^i - \xi_{2,i}\|_\infty \leq C h_i$.

Following [FI96] we now set

$$Bv = B_1 \underbrace{-2s(\Delta_\xi \varphi) v}_{= B_2 v}, \qquad g = g' - 2s(\Delta_{t,x} \varphi) v.$$

An explanation for the introduction of this additional term $B_2 v$ is provided in [LL09].

Equation (3.5) now reads $Av + Bv = g$ and we write

$$\|Av\|^2_{L^2(Q)} + \|Bv\|^2_{L^2(Q)} + 2 \operatorname{Re} (Av, Bv)_{L^2(Q)} = \|g\|^2_{L^2(Q)}. \tag{3.6}$$

We shall need the following estimation of $\|g\|_{L^2(Q)}$. The proof can be adapted from the one-dimensional case (see Lemma 4.2 and its proof in [BHL09a]).

Lemma 3.2 (Estimate of the r.h.s.). *For $sh \leq \mathfrak{K}$ we have*

$$\|g\|^2_{L^2(Q)} \leq C_{\lambda, \mathfrak{K}} \left(\|rf\|^2_{L^2(Q)} + s^2 \|v\|^2_{L^2(Q)} + (sh)^2 \sum_{i \in [\![1,d]\!]} \|D_i v\|^2_{L^2(Q)} \right). \tag{3.7}$$

Most of the remaining of the proof will be dedicated to computing the inner-product $\operatorname{Re} (Av, Bv)_{L^2(Q)}$. Developing this term, we set $I_{ij} = \operatorname{Re} (A_i v, B_j v)_{L^2(Q)}$.

Lemma 3.3 (Estimate of I_{11}). *For $sh \leq \mathfrak{K}$, the term I_{11} can be estimated from* *below in the following way*

$$
\begin{aligned}
I_{11} \geq & -s\lambda^2\Big(\|\varphi^{\frac{1}{2}}|\nabla_\xi\psi|\partial_t v\|^2_{L^2(Q)} + \|\varphi^{\frac{1}{2}}|\nabla_\xi\psi|\Upsilon_\xi v\|^2_{L^2(Q)}\Big)\\
& + s\lambda \iint_\Omega \Big(\varphi(\partial_t\psi)|\Upsilon_\xi v|^2\Big)(\mathcal{T}) - s\lambda\Big[\iint_\Omega \varphi(\partial_t\psi)|\partial_t v|^2\Big]_0^{\mathcal{T}}\\
& + Y_{11} - X_{11} - W_{11} - J_{11},
\end{aligned}
$$

with

$$
\begin{aligned}
Y_{11} = \sum_{i\in[\![1,d]\!]} \iint_{Q_i} \Big(& \big((\xi_{1,i}^2\xi_{2,i}^2 + \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2))\, r\overline{D_i\rho}^i\big)_{|k_i=N_i+1}|D_i v|^2_{|k_i=N_i+\frac{1}{2}}\\
& - \big((\xi_{1,i}^2\xi_{2,i}^2 + \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2))\, r\overline{D_i\rho}^i\big)_{|k_i=0}|D_i v|^2_{|k_i=\frac{1}{2}}\Big)\, dt,
\end{aligned}
$$

and

$$
X_{11} = \iiint_Q \beta_{11}|\partial_t v|^2\, dt + \sum_{i\in[\![1,d]\!]}\iiint_Q \nu_{11,i}|D_i v|^2\, dt + \sum_{i\in[\![1,d]\!]}\iiint_Q \overline{\nu}_{11,i}|\overline{D_i v}^i|^2\, dt,
$$

with β_{11}, $\nu_{11,i}$, $\overline{\nu}_{11,i}$ *of the form* $s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ *and*

$$
W_{11} = \iiint_Q \gamma_{11,it}|D_i\partial_t v|^2\, dt + \sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}}\iiint_Q \gamma_{11,ij}|D_iD_j v|^2\, dt + \sum_{i\in[\![1,d]\!]}\iiint_Q \gamma_{11,ii}|\bar{D}_iD_i v|^2\, dt,
$$

with $\gamma_{11,it}$, $\gamma_{11,ij}$, *and* $\gamma_{11,ii}$ *of the form* $h^2\big(s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)$ *and*

$$
\begin{aligned}
J_{11} = & \sum_{i\in[\![1,d]\!]}\iint_\Omega \delta_{11,i}|D_i v|^2(\mathcal{T})\\
& + \sum_{i\in[\![1,d]\!]}\iint_{Q_i}\Big((\delta_{11,i}^{(2)})_{|k_i=N_i+\frac{1}{2}}|D_i v|^2_{|k_i=N_i+\frac{1}{2}} + (\delta_{11,i}^{(2)})_{|k_i=\frac{1}{2}}|D_i v|^2_{|k_i=\frac{1}{2}}\Big)\, dt,
\end{aligned}
$$

with $\delta_{11,i} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$, *and* $\delta_{11,i}^{(2)} = sh_i\lambda\varphi\mathcal{O}(1) + sh_i\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$. The proof can be found in Appendix C.

The following lemma can be readily adapted from its counterpart in [BHL09a, Lemma 4.4] (use also Lemma 4.8 in [BHL09a]).

LEMMA 3.4 (Estimate of I_{12}). *For* $sh \leq \mathfrak{K}$, *the term* I_{12} *is of the following form*

$$
I_{12} \geq 2s\lambda^2\Big(\|\varphi^{\frac{1}{2}}|\nabla_\xi\psi|\partial_t v\|^2_{L^2(Q)} + \|\varphi^{\frac{1}{2}}|\nabla_\xi\psi|\Upsilon_\xi v\|^2_{L^2(Q)}\Big) - X_{12} - J_{12},
$$

with

$$
\begin{aligned}
X_{12} &= \iiint_Q \beta_{12}|\partial_t v|^2\ dt + \sum_{i\in[\![1,d]\!]}\iiint_Q \nu_{12,i}|D_i v|^2\ dt + \iiint_Q \mu_{12}|v|^2\ dt,\\
J_{12} &= \iint_\Omega \eta_{12}|v|^2(\mathcal{T}) + \iint_\Omega \mathcal{O}(1)|\partial_t v|^2(\mathcal{T}),
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_{12} = s\lambda\varphi\mathcal{O}(1), \quad \mu_{12} = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1), \quad \eta_{12} = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\
\nu_{12,i} = s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh).
\end{aligned}
$$

Lemma 3.5 (Estimate of I_{21}). *For $sh \leq \mathfrak{K}$, the term I_{21} can be estimated from below in the following way*

$$I_{21} \geq 3s^3\lambda^4 \|\varphi^{\frac{3}{2}} |\nabla_\xi \psi|^2 v\|^2_{L^2(Q)} - (s\lambda)^3 \iint_\Omega (\varphi^3(\partial_t\psi)|\nabla_\xi\psi|^2)(\mathcal{T})\,|v|^2(\mathcal{T})$$
$$+ Y_{21} - W_{21} - X_{21} - J_{21},$$

with

$$W_{21} = \sum_{i\in[\![1,d]\!]} \iiint_Q \gamma_{21,it}|D_i\partial_t v|^2\,dt + \sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \iiint_Q \gamma_{21,ij}|D_iD_jv|^2\,dt,$$
$$Y_{21} = \sum_{i\in[\![1,d]\!]} \iint_{Q_i} \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)(r\overline{D_i\rho}^{\,i})_0|D_iv|^2_{|\frac{1}{2}}\,dt$$
$$+ \sum_{i\in[\![1,d]\!]} \iint_{Q_i} \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)(r\overline{D_i\rho}^{\,i})_{N_x+1}|D_iv|^2_{|N_x+\frac{1}{2}}\,dt,$$
$$X_{21} = \iiint_Q \mu_{21}|v|^2\,dt + \sum_{i\in[\![1,d]\!]} \iiint_Q \nu_{21,i}|D_iv|^2\,dt$$
$$J_{21} = \iint_\Omega \eta_{21}|v|^2(\mathcal{T}) + \sum_{i\in[\![1,d]\!]} \iint_\Omega \delta_{21,i}|D_iv|^2(\mathcal{T}),$$

where

$$\gamma_{21,it} = h\mathcal{O}(sh), \quad \gamma_{21,ij} = h\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2),$$
$$\mu_{21} = (s\lambda\varphi)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \quad \nu_{21,i} = s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2),$$
$$\eta_{21} = s^3\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1), \quad \textit{and} \;\; \delta_{21,i} = s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2).$$

The proof can be found in Appendix C.

The following lemma can be readily adapted from its counterpart in [BHL09a, Lemma 4.6].

Lemma 3.6 (Estimate of I_{22}). *For $sh \leq \mathfrak{K}$, the term I_{22} is of the following form*

$$I_{22} = -2s^3\lambda^4\|\varphi^{\frac{3}{2}}|\nabla_\xi\psi|^2v\|^2_{L^2(Q)} - X_{22},$$

with

$$X_{22} = \iiint_Q \mu_{22}|v|^2\;dt + \sum_{i\in[\![1,d]\!]} \iiint_Q \nu_{22,i}|D_iv|^2\;dt$$

where $\mu_{22} = (s\lambda\varphi)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$, and $\nu_{22,i} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$.

Continuation of the proof of Theorem 3.1. Collecting the terms we have obtained in the previous lemmata, from (3.6) we obtain, for $sh \leq \mathfrak{K}$,

$$\begin{aligned}
&2s^3\lambda^4\|\varphi^{\frac{3}{2}}|\nabla_\xi\psi|^2v\|^2_{L^2(Q)} + 2s\lambda^2\Big(\|\varphi^{\frac{1}{2}}|\nabla_\xi\psi|\partial_tv\|^2_{L^2(Q)} + \|\varphi^{\frac{1}{2}}|\nabla_\xi\psi|\Upsilon_\xi v\|^2_{L^2(Q)}\Big)\\
&\quad + 2s\lambda\Big(\sum_{i\in[\![1,d]\!]}\iint_\Omega \xi_{1,i}\xi_{2,i}(\varphi\partial_t\psi)(\mathcal{T})\,|D_iv|^2(\mathcal{T}) - \Big[\iint_\Omega \varphi(\partial_t\psi)\,|\partial_tv|^2\Big]_0^{\mathcal{T}}\Big)\\
&- 2(s\lambda)^3\iint_\Omega(\varphi^3(\partial_t\psi)|\nabla_\xi\psi|^2)(\mathcal{T})\,|v|^2(\mathcal{T}) + 2Y \leq C_{\lambda,\mathfrak{K}}\|rf\|^2_{L^2(Q)} + 2X + 2W + 2J,
\end{aligned} \tag{3.8}$$

where $Y = Y_{11}+Y_{21}$, $X = X_{11}+X_{12}+X_{21}+X_{22}+C_{\lambda,\mathfrak{K}}\Big(s^2\|v\|^2_{L^2(Q)}+(sh)^2\sum_{i\in[\![1,d]\!]}\|D_iv\|^2_{L^2(Q)}\Big)$, $W = W_{11} + W_{21}$, and $J = J_{11} + J_{12} + J_{21}$.

With the following lemma, we may in fact ignore the term Y.

Lemma 3.7. *Let $sh \leq \mathfrak{K}$. For all λ there exists $\varepsilon_1(\lambda) > 0$ such that for $0 < sh \leq \varepsilon_1(\lambda)$, we have $Y \geq 0$.*

As $|\nabla_\xi\psi| \geq C > 0$ in Q and recall the properties of the coefficients $\xi_{1,i}$ and $\xi_{2,i}$ we then have

$$2s^3\lambda^4\|\varphi^{\frac{3}{2}}v\|^2_{L^2(Q)} + 2s\lambda^2\Big(\|\varphi^{\frac{1}{2}}\partial_tv\|^2_{L^2(Q)} + \|\varphi^{\frac{1}{2}}\Upsilon v\|^2_{L^2(Q)}\Big)$$
$$+ 2s\lambda\Big(\sum_{i\in[\![1,d]\!]}\iint_\Omega \xi_{1,i}\xi_{2,i}(\varphi\partial_t\psi)(\mathcal{T})\,|D_iv|^2(\mathcal{T}) - \Big[\iint_\Omega \varphi(\partial_t\psi)\,|\partial_tv|^2\Big]_0^{\mathcal{T}}\Big)$$
$$- 2(s\lambda)^3\iint_\Omega(\varphi^3(\partial_t\psi)|\nabla_\xi\psi|^2)(\mathcal{T})\,|v|^2(\mathcal{T})+ \leq C_{\lambda,\mathfrak{K}}\|rf\|^2_{L^2(Q)} + 2X + 2W + 2J, \quad (3.9)$$

Lemma 3.8. *We have*
$$s\lambda^2\Big(\|\varphi^{\frac{1}{2}}\partial_tv\|^2_{L^2(Q)} + \|\varphi^{\frac{1}{2}}\Upsilon v\|^2_{L^2(Q)}\Big) \geq \nu(h,\lambda) + CH - \tilde{X} - \tilde{W},$$
where $\nu(h,\lambda) \geq 0$ for $0 < h \leq h_1(\lambda)$ for some $h_1(\lambda)$ sufficiently small and
$$H = s\lambda^2\sum_{i\in[\![1,d]\!]}\iiint_Q \varphi\big|\overline{D_iv}^i\big|^2\,dt + s\lambda^2h^2\Big(\sum_{i\in[\![1,d]\!]}\iiint_Q \varphi|D_i\partial_tv|^2\,dt$$
$$+ \sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}}\iiint_Q \varphi|D_iD_jv|^2\,dt + \sum_{i\in[\![1,d]\!]}\iiint_Q \varphi|\bar{D}_iD_iv|^2\,dt\Big),$$

$$\tilde{X} = sh^2\Big(\iiint_Q \mathcal{O}_\lambda(1)\big|\partial_tv\big|^2\,dt + \sum_{i\in[\![1,d]\!]}\iiint_Q \mathcal{O}_\lambda(1)|D_iv|^2\,dt$$
$$+ \sum_{i\in[\![1,d]\!]}\iiint_Q \mathcal{O}_\lambda(1)\big|\overline{D_iv}^i\big|^2\,dt\Big),$$

and
$$\tilde{W} = sh^4\Big(\sum_{i\in[\![1,d]\!]}\iiint_Q \mathcal{O}_\lambda(1)|\partial_tD_iv|^2\,dt$$
$$+ \sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}}\iiint_Q \mathcal{O}_\lambda(1)|D_iD_jv|^2\,dt + \sum_{i\in[\![1,d]\!]}\iiint_Q \mathcal{O}_\lambda(1)|\bar{D}_iD_iv|^2\,dt\Big).$$

End of the proof of Theorem 3.1. Recalling the properties satisfied by ψ listed in Assumption 1.3, if we choose $\lambda_1 \geq 1$ sufficiently large, then for $\lambda = \lambda_1$ (fixed for the rest of the proof) and $sh \leq \varepsilon_1(\lambda_1)$ and $0 < h \leq h_1(\lambda_1)$, from (3.8) and Lemmata 3.7 and 3.8, we obtain

$$s^3\|v\|^2_{L^2(Q)} + s\|\partial_tv\|^2_{L^2(Q)} + s\sum_{i\in[\![1,d]\!]}\|D_iv\|^2_{L^2(Q)} + \underline{H}$$
$$+ s|\partial_tv(0,.)|^2_{L^2(\Omega)} + s|\partial_tv(\mathcal{T},.)|^2_{L^2(\Omega)} + s^3|v(\mathcal{T},.)|^2_{L^2(\Omega)}$$
$$\leq C_{\lambda_1,\mathfrak{K}}\Big(\|rf\|^2_{L^2(Q)} + s\sum_{i\in[\![1,d]\!]}|D_iv(\mathcal{T},.)|^2_{L^2(\Omega)} + s|\partial_tv(0,.)|^2_{L^2(\omega)}\Big)$$
$$+ \underline{X} + \underline{W} + \underline{J}, \quad (3.10)$$

where

$$\underline{H} = s \sum_{i\in[\![1,d]\!]} \|\overline{D_i v}^i\|^2_{L^2(Q)} + sh^2\Big(\sum_{i\in[\![1,d]\!]} \|D_i \partial_t v\|^2_{L^2(Q)} + \sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \|D_i D_j v\|^2_{L^2(Q)} + \sum_{i\in[\![1,d]\!]} \|\bar{D}_i D_i v\|^2_{L^2(Q)}\Big),$$

$$\underline{X} = \iiint_Q \mu_1 |v|^2\, dt + \sum_{i\in[\![1,d]\!]} \iiint_Q \nu_{1,i} |D_i v|^2\, dt + \sum_{i\in[\![1,d]\!]} \iiint_Q \overline{\nu}_{1,i} |\overline{D_i v}^i|^2\, dt + \iiint_Q \beta_1 |\partial_t v|^2\, dt,$$

with $\mu_1 = s^2\mathcal{O}_{\lambda_1,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda_1,\mathfrak{K}}(sh)$ and $\nu_{1,i}$, $\overline{\nu}_{1,i}$, β_1, all of the form $s\mathcal{O}_{\lambda_1,\mathfrak{K}}(sh)$, and where

$$\underline{W} = \sum_{i\in[\![1,d]\!]} \iiint_Q \gamma_{1,it} |D_i\partial_t v|^2\, dt + \sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \iiint_Q \gamma_{1,ij} |D_i D_j v|^2\, dt + \sum_{i\in[\![1,d]\!]} \iiint_Q \gamma_{1,ii} |\bar{D}_i D_i v|^2\, dt,$$

where $\gamma_{1,it}$, $\gamma_{1,ij}$ and $\gamma_{1,ii}$ are of the form $sh^2\mathcal{O}_{\lambda_1,\mathfrak{K}}(sh)$, and where

$$\underline{J} = \iint_\Omega \eta_1 |v|^2(\mathcal{T}) + \sum_{i\in[\![1,d]\!]} \iint_\Omega \delta_{1,i} |D_i v|^2(\mathcal{T}) + \sum_{i\in[\![1,d]\!]} \iint_{Q_i} \Big((\delta^{(2)}_{1,i})_{N_i+\frac12} |D_i v|^2_{N_i+\frac12} + (\delta^{(2)}_{1,i})_{\frac12} |D_i v|^2_{\frac12}\Big)\, dt,$$

with $\eta_1 = s^3\mathcal{O}_{\lambda_1,\mathfrak{K}}(sh) + s^2\mathcal{O}_{\lambda_1,\mathfrak{K}}(1)$ and $\delta_{1,i} = s\mathcal{O}_{\lambda_1,\mathfrak{K}}(sh)$, $\delta^{(2)}_{1,i} = sh_i\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$. The last term in $\underline{J}$ was obtained by "absorbing" the following term in J_{11}

$$s\lambda \sum_{i\in[\![1,d]\!]} \iint_{Q_i} h_i\Big((\varphi)_{N_i+\frac12}\mathcal{O}(1)|D_i v|^2_{N_i+\frac12} + (\varphi)_{\frac12}\mathcal{O}(1)|D_i v|^2_{\frac12}\Big)\, dt,$$

by the volume term

$$s\lambda^2 \sum_{i\in[\![1,d]\!]} \iiint_Q \xi_{1,i}\xi_{2,i}\varphi |\nabla_\xi \psi|^2 |D_i v|^2\ dt,$$

for λ large.

We can now choose ε_0 and h_0 sufficiently small, with $0 < \varepsilon_0 \le \varepsilon_1(\lambda_1)$, $0 < h_0 \le h_1(\lambda_1)$, and $s_0 \ge 1$ sufficiently large, such that for $s \ge s_0$, $0 < h \le h_0$, and $sh \le \varepsilon_0$, we obtain

$$\begin{aligned} &s^3\|v\|^2_{L^2(Q)} + s\|\partial_t v\|^2_{L^2(Q)} + s\sum_{i\in[\![1,d]\!]} \|D_i v\|^2_{L^2(Q)} + \underline{H} \\ &\qquad + s|\partial_t v(0,.)|^2_{L^2(\Omega)} + s|\partial_t v(\mathcal{T},.)|^2_{L^2(\Omega)} + s^3|v(\mathcal{T},.)|^2_{L^2(\Omega)} \\ &\le C_{\lambda_1,\mathfrak{K},\varepsilon_0,s_0}\Big(\|rf\|^2_{L^2(Q)} + s\sum_{i\in[\![1,d]\!]} |D_i v(\mathcal{T},.)|^2_{L^2(\Omega)} + s|\partial_t v(0,.)|^2_{L^2(\omega)}\Big). \end{aligned} \tag{3.11}$$

To finish the proof, we need to express all the terms in the estimate above in terms of the original function u. We can proceed exactly as in the end of proof of Theorem 4.1 in [BHL09a]. ☐

4. Carleman estimates for non uniform meshes. We consider here the notation introduced in section 1.1.8.

We define, for $i \in [\![1,d]\!]$, $\zeta_i \in \mathbb{C}^{\overline{\mathfrak{M}}^i}$ and $\bar{\zeta}_i \in \mathbb{C}^{\mathfrak{M}}$ as follows

$$\zeta_{i,\boldsymbol{k}} = \frac{h_{i,k_i}}{h_i^\star}, \quad \boldsymbol{k} \in \overline{\mathfrak{N}}^i, \qquad \bar{\zeta}_{i,\boldsymbol{k}} = \frac{h_{i,k_i}}{h_i^\star}, \quad \boldsymbol{k} \in \mathfrak{N}.$$

Even though these two formulae look similar they are in fact different as the indices $\boldsymbol{k}$ are taken in different sets.

Lemma 4.1. *We have the following properties*

$$\mathrm{reg}(\vartheta)^{-1} \leq \zeta_{i,\boldsymbol{k}} \leq \mathrm{reg}(\vartheta), \quad i \in [\![1,d]\!], \boldsymbol{k} \in \overline{\mathfrak{N}}^i,$$

$$\mathrm{reg}(\vartheta)^{-1} \leq \bar{\zeta}_{i,\boldsymbol{k}} \leq \mathrm{reg}(\vartheta), \quad i \in [\![1,d]\!], \boldsymbol{k} \in \mathfrak{N},$$

$$|\bar{D}_i \zeta_i|_{L^\infty(\Omega)} \leq \mathrm{reg}(\vartheta)^2, \quad \text{and} \quad |D_i \bar{\zeta}_i|_{L^\infty(\Omega)} \leq \mathrm{reg}(\vartheta)^2.$$

For $u \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$, we define $\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0} u \in \mathbb{C}^{\mathfrak{M}_0\cup\partial\mathfrak{M}_0}$ to be the discrete function corresponding to the reference uniform mesh $\mathfrak{M}_0$ which takes the same values as u for each index $\boldsymbol{k} \in \mathfrak{N}$. Similarly, for $i \in [\![1,d]\!]$ and $u \in \mathbb{C}^{\overline{\mathfrak{M}}^i}$, we denote by $\mathcal{Q}_{\overline{\mathfrak{M}}^i}^{\overline{\mathfrak{M}_0}^i} u \in \mathbb{C}^{\overline{\mathfrak{M}_0}^i}$ the discrete function defined on $\overline{\mathfrak{M}_0}^i$ which takes the same values as u for each index $\boldsymbol{k} \in \overline{\mathfrak{N}}^i$. We denote by $\mathcal{Q}_{\mathfrak{M}_0}^{\mathfrak{M}}$ and $\mathcal{Q}_{\overline{\mathfrak{M}}^i}^{\overline{\mathfrak{M}_0}^i}$ the inverse of the operators $\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}$ and $\mathcal{Q}_{\overline{\mathfrak{M}_0}^i}^{\overline{\mathfrak{M}}^i}$ respectively.

Lemma 4.2.

- *For any $i \in [\![1,d]\!]$, any $u \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ and any $v \in \mathbb{C}^{\overline{\mathfrak{M}}^i}$, we have*

$$D_i(\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0} u) = \mathcal{Q}_{\overline{\mathfrak{M}}^i}^{\overline{\mathfrak{M}_0}^i}(\zeta_i D_i u), \quad \bar{D}_i(\mathcal{Q}_{\overline{\mathfrak{M}}^i}^{\overline{\mathfrak{M}_0}^i} v) = \mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}(\bar{\zeta}_i \bar{D}_i v).$$

- *For any $u \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ and any $i \in [\![1,d]\!]$, we have*

$$\bar{D}_i(\gamma_i D_i u) = (\bar{\zeta}_i)^{-1} \mathcal{Q}_{\mathfrak{M}_0}^{\mathfrak{M}} \left(\bar{D}_i \left(\left(\mathcal{Q}_{\overline{\mathfrak{M}}^i}^{\overline{\mathfrak{M}_0}^i} \frac{\gamma_i}{\zeta_i} \right) D_i (\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0} u) \right) \right).$$

Lemma 4.3. *For any $u \in \mathbb{C}^{\mathfrak{M}}$, and any $v \in \mathbb{C}^{\overline{\mathfrak{M}}^i}$, $i \in [\![1,d]\!]$, we have*

$$\mathrm{reg}(\vartheta)^{-1} |u|_{L^2(\Omega)}^2 \leq |\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0} u|_{L^2(\Omega^\star)}^2 \leq \mathrm{reg}(\vartheta) |u|_{L^2(\Omega)}^2,$$

$$\mathrm{reg}(\vartheta)^{-1} |v|_{L^2(\Omega)}^2 \leq |\mathcal{Q}_{\overline{\mathfrak{M}}^i}^{\overline{\mathfrak{M}_0}^i} v|_{L^2(\Omega^\star)}^2 \leq \mathrm{reg}(\vartheta) |v|_{L^2(\Omega)}^2.$$

We can now prove the Carleman estimate of Theorem 1.4 for the semi-discrete elliptic operator

$$P^{\mathfrak{M}} = -\partial_t^2 - \sum_{i \in [\![1,d]\!]} \bar{D}_i(\gamma_i D_i \cdot).$$

We only give a sketch of the proof, since it is very similar to the one which is detailed in [BHL09a] for the one-dimensional case.

Proof of Theorem 1.4. The key idea is to perform a change of variables that transforms $P^{\mathfrak{M}}$ defined on a non-uniform mesh into an semi-discrete elliptic operator defined on a uniform mesh. All the geometric information concerning the initial mesh is then contained in the coefficients of this new operator.

More precisely, we consider the discrete function $w = \mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0} u$ which is defined on the uniform mesh $\mathfrak{M}_0$. By using Lemma 4.2 we observe that

$$\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}(P^{\mathfrak{M}} u) = -\partial_t^2 w - \sum_{i \in [\![1,d]\!]} \left(\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}(\bar{\zeta}_i)^{-1} \right) \left(\bar{D}_i \left(\left(\mathcal{Q}_{\overline{\mathfrak{M}}^i}^{\overline{\mathfrak{M}_0}^i} \frac{\gamma_i}{\zeta_i} \right) D_i w \right) \right).$$

We introduce the operator $P^{\mathfrak{M}_0} = -\partial_t^2 - \sum_{i \in [\![1,d]\!]} \xi_{1,i} \left(\bar{D}_i \left(\xi_{2,i} D_i w \right) \right)$ with

$$\xi_{1,i} = \mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}(\bar{\zeta}_i)^{-1}, \quad \xi_{2,i} = \mathcal{Q}_{\overline{\mathfrak{M}}^i}^{\overline{\mathfrak{M}_0}^i} \frac{\gamma_i}{\zeta_i},$$

so that we may now apply the Carleman estimate of Theorem 3.1 to w and $P^{\mathfrak{M}_0}$ on the uniform mesh $\mathfrak{M}_0$ and with the weight function $\boldsymbol{x} \in [0,1]^d \mapsto \psi \circ (\vartheta_1(x_1) \ldots \vartheta_d(x_d))$.

We note that $\operatorname{reg}(\xi)$ is bounded by some constant depending only on $\operatorname{reg}(\vartheta)$ and $\operatorname{reg}(\Gamma)$ and independent of the size of the mesh. We can thus find reg^0 sufficiently large for which Theorem 3.1 leads to a Carleman inequality for the function w, and the weight function defined above.

Using Lemmata 4.2 and 4.3 we then deduce result. Note that the values of h_0, ε_0, may change, depending only on the values of $\operatorname{reg}(\vartheta)$ and $\operatorname{reg}(\Gamma)$ and not on the mesh size.
□

Appendix A. Construction of a weight function.

A weight function that satisfies the conditions listed in Assumption 1.3 can be constructed as follows.

We first start with a function $\phi_1 \in \mathscr{C}^\infty([0,\mathcal{T}])$ such that $\partial_t \phi_1(0) \geq C > 0$, $\partial_t \phi_1(\mathcal{T}) \leq -C < 0$, and $\phi_1(0) = \phi_1(\mathcal{T}) = 0$, and $\phi_1(t) > 0$ if $t \in (0,\mathcal{T})$. We choose ϕ_1 with a single critical point.

Let also $\phi_2 \in \mathscr{C}^\infty(\bar{\bar{\Omega}})$ be such that $\phi_2 \geq C > 0$ and $\partial_{n_x} \phi_2 \leq -C' < 0$ and $\partial_i^2 \phi_2 \geq C'' > 0$ in $V_{\partial\Omega}$.

This can be achieved with $\phi_2(x) = e^{\zeta \tilde{\phi}_2(x)} + C - 1$, with $\tilde{\phi}_2 = 0$ on $\partial\Omega$,

$$\tilde{\phi}_2 > 0, \text{ in } \Omega, \quad \tilde{\phi}_2 = 0 \text{ and } \partial_{n_x} \tilde{\phi}_2 \leq -\tilde{C} < 0, \text{ on } \partial\Omega$$

and $\zeta > 0$ sufficiently large and by taking the neighborhood $V_{\partial\Omega}$ sufficiently small. The function ϕ_2 can be chosen with a finite number of critical points by means of Morse theorem [AE84].

We next set $\phi(t,\boldsymbol{x}) = \phi_1(t)\phi_2(\boldsymbol{x})$. This function satisfies the desired properties listed in Assumption 1.3 on the boundaries $(0,\mathcal{T}) \times \partial\Omega$ (and in its neighborhood $(0,\mathcal{T}) \times V_{\partial\Omega}$), $\{0\} \times (\Omega \setminus \omega)$ and $\{\mathcal{T}\} \times \Omega$. It is also characterized by a finite number of critical points.

We choose y_0 in $\{0\} \times \omega$. We enlarge Q in a small neighborhood of y_0 which leaves ∂Q unchanged outside of $\{0\} \times \omega$. We call $\mathcal{Q}$ this extension of Q and we extend the function ϕ to $\mathcal{Q}$ in a $\mathscr{C}^k$ manner. The critical points of ϕ can be pulled back to the interior of $\mathcal{Q} \setminus Q$ by composing ϕ with a finite number of diffeomorphisms (see [FI96] for the construction of these diffeomorphisms). The resulting function is the weight function ψ and it satisfies all the properties listed in Assumption 1.3.

Appendix B. Proofs of some technical results in Section 2.

B.1. Proof of Proposition 2.9. We recall that $r\rho = 1$. By Lemma 2.6 we have $\tau_i^+\partial^\alpha\rho(\boldsymbol{x}) = \partial^\alpha\rho(\boldsymbol{x}) + Ch_i\rho(\boldsymbol{x})\int_0^1 r(x)\partial_i\partial^\alpha\rho(\boldsymbol{x}+\sigma\boldsymbol{h}_i/2)\, d\sigma$, which by Lemma 2.7 yields $r\tau_i^+\partial^\alpha\rho = r\partial^\alpha\rho + s^{|\alpha|}\mathcal{O}_\lambda(sh)e^{\mathcal{O}_\lambda(sh)} = s^{|\alpha|}\mathcal{O}_{\lambda,\mathfrak{K}}(1)$. The proof is the same for $r\tau_i^-\partial^\alpha\rho$. For $r\mathsf{D}_i\rho$, $r\mathsf{A}_i\partial^\alpha\rho = r\widehat{\partial^\alpha\rho}^{\,i}$, $r\mathsf{A}_i^2\partial^\alpha\rho = r\widehat{\widehat{\partial^\alpha\rho}}^{\,i}$, and $r\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\rho$ we proceed similarly, exploiting the formula in Lemma 2.6 and then applying the result of Lemma 2.7, e.g.,

$$
\begin{aligned}
\mathsf{D}_i\rho(\boldsymbol{x}) &= \partial_i\rho(\boldsymbol{x}) + Ch_i^2\rho(\boldsymbol{x})\int_{-1}^{1}(1-|\sigma|)^2 r(x)(\partial_i^3\rho)(\boldsymbol{x}+\sigma\boldsymbol{h}_i/2)\, d\sigma\\
&= \partial_i\rho(\boldsymbol{x}) + s\rho(\boldsymbol{x})\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) = sr(\boldsymbol{x})\mathcal{O}_{\lambda,\mathfrak{K}}(1).
\end{aligned}
$$

Noting that $\mathsf{A}_i\mathsf{D}_i\rho(\boldsymbol{x}) = \widehat{\mathsf{D}_i\rho}^{\,i}(\boldsymbol{x}) = (2h_i)^{-1}(\rho(\boldsymbol{x}+\boldsymbol{h}_i) - \rho(\boldsymbol{x}-\boldsymbol{h}_i))$ we proceed as we did for $\mathsf{D}_i r$. □

B.2. Proof of Lemma 2.10. By Lemma 2.6, we write

$$
\mathsf{D}_i(\partial^\beta(r\partial^\alpha\rho))(\boldsymbol{x}) = \partial_i\partial^\beta(r\partial^\alpha\rho)(\boldsymbol{x}) + Ch_i^2\int_{-1}^{1}(1-|\sigma|)^2\partial_i^3\partial^\beta(r\partial^\alpha\rho)(\boldsymbol{x}+\sigma\boldsymbol{h}_i/2)\, d\sigma.
$$

By Lemma 2.7 we have $\partial_i^3\partial^\beta(r\partial^\alpha\rho) = \mathcal{O}_\lambda(s^{|\alpha|})$, which yields the first result in the case $k_i + k_j = 1$. For the case $k_i + k_j = 2$, we proceed similarly, making use of the other formulae listed in Lemma 2.6. For the averaging cases, we make use of the second formula in Lemma 2.6.

Following the proof of Lemma 2.7 in [BHL09a] we set $\nu(\boldsymbol{x},\sigma\boldsymbol{h}_i) := r(\boldsymbol{x})\rho(\boldsymbol{x}+\sigma\boldsymbol{h}_i)$. We have

$$
\mathsf{D}_i\partial^{\beta'}\nu(\boldsymbol{x},\sigma\boldsymbol{h}_i) = \frac{1}{2}\int_{-1}^{1}(\partial_i\partial^{\beta'}\nu)(\boldsymbol{x}+\sigma'\boldsymbol{h}_i/2,\sigma\boldsymbol{h}_i)\, d\sigma' = \mathcal{O}_{\lambda,\mathfrak{K}}(1), \quad \text{for } |\beta'| \leq |\beta|, \tag{B.1}
$$

for $sh \leq \mathfrak{K}$ by Lemma 2.7. Next, with $\mu_\alpha = r\partial^\alpha\rho$, we write $r(\boldsymbol{x})\partial^\alpha\rho(\boldsymbol{x}+\sigma\boldsymbol{h}_i) = \nu(\boldsymbol{x},\sigma\boldsymbol{h}_i)\mu_\alpha(\boldsymbol{x}+\sigma\boldsymbol{h}_i)$, which gives $\mathsf{D}_i\partial^\beta(r(\boldsymbol{x})\partial^\alpha\rho(\boldsymbol{x}+\sigma\boldsymbol{h}_i))$ as a linear combination of terms of the form

$$
\mathsf{A}_i(\partial^{\beta'}\nu(.,\sigma\boldsymbol{h}_i))\,\mathsf{D}_i(\partial^{\beta''}\mu_\alpha(.+\sigma\boldsymbol{h}_i)) + \mathsf{D}_i(\partial^{\beta'}\nu(.,\sigma\boldsymbol{h}_i))\,\mathsf{A}_i(\partial^{\beta''}\mu_\alpha(.+\sigma\boldsymbol{h}_i)), \quad \beta'+\beta''=\beta,
$$

by the continuous and discrete Leibniz rules (Lemma 2.1). By the first part and Lemma 2.7 we have $\mathsf{D}_i(\partial^{\beta''}\mu_\alpha(\boldsymbol{x}+\sigma\boldsymbol{h}_i)) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|})$. By Lemma 2.7, $\partial^{\beta'}\nu(\boldsymbol{x},\sigma\boldsymbol{h}_i) = \mathcal{O}_{\lambda,\mathfrak{K}}(1)$ and $\partial^{\beta''}\mu_\alpha(\boldsymbol{x}+\sigma\boldsymbol{h}_i) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|})$. The last result hence follows from (B.1). We proceed in a similar way for the case $k_i + k_j = 2$. □

B.3. Proof of Lemma 2.11. For the first two results, we proceed as in Lemma 2.10 and use Corollary 2.8.

For the last results we use the continuous and discrete Leibniz rules (Lemma 2.1) and Lemma 2.10. □

B.4. Proof of Proposition 2.12. Taylor formulae yield

$$
\widehat{\mathsf{D}_i\rho}^{\,i}(\boldsymbol{x}) = \frac{\rho(\boldsymbol{x}+\boldsymbol{h}_i) - \rho(\boldsymbol{x}-\boldsymbol{h}_i)}{2h_i} = \partial_i\rho(\boldsymbol{x}) + Ch_i^2\int_{-1}^{1}(1-|\sigma|)^2\partial_i^3\rho(\boldsymbol{x}+\sigma\boldsymbol{h}_i)\, d\sigma, \tag{B.2}
$$

which in turn gives

$$\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\mathsf{A}_i^k\partial^\alpha(r\widehat{\mathsf{D}_i\rho}^{\,i}))(\boldsymbol{x}) = \mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\mathsf{A}_i^k\partial^\alpha(r\partial_i\rho)(\boldsymbol{x}) \\ + Ch_i^2\int_{-1}^{1}(1-|\sigma|)^2\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\mathsf{A}_i^k\partial^\alpha(r(\boldsymbol{x})\partial_i^3\rho(\boldsymbol{x}+\sigma\boldsymbol{h}_i))\,d\sigma,$$

and the first result follows by Lemma 2.10 (and Lemma 2.7 for the second equality).

Next, from Lemma 2.6, we write

$$\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}(r\mathsf{D}_i^2\rho)(\boldsymbol{x}) = \mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}(r\partial_i^2\rho)(\boldsymbol{x}) \\ + Ch_i^2\int_{-1}^{1}(1-|\sigma|)^3\;\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}(r(\boldsymbol{x})\partial_i^4\rho(\boldsymbol{x}+\sigma\boldsymbol{h}_i))\,d\sigma,$$

and the third result follows as above. For $\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}(r\mathsf{A}^2\rho)$ we use the formula for $\mathsf{A}^2\rho$ given in Lemma 2.6 and proceed as above. ∎

B.5. Proof of Proposition 2.13. From (B.2) we write

$$\mathsf{A}_i^{k_i'}\mathsf{A}_j^{k_j'}\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\beta\left(r^2(\partial^\alpha\rho)\widehat{\mathsf{D}_i\rho}^{\,i}\right) = \mathsf{A}_i^{k_i'}\mathsf{A}_j^{k_j'}\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\beta\left(r^2(\partial^\alpha\rho)\partial_i\rho\right) \\ + Ch_i^2\int_{-1}^{1}(1-|\sigma|)^2\mathsf{A}_i^{k_i'}\mathsf{A}_j^{k_j'}\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\beta\left(r^2(\partial^\alpha\rho)\partial_i^3\rho(.+\sigma\boldsymbol{h}_i)\right)\,d\sigma,$$

and we conclude with Lemma 2.11. For the next two results we use the formulae listed in Lemma 2.6 and proceed as above.

From Lemma 2.6, equation (B.2), and by Lemma 2.11 we have

$$\begin{aligned}\mathsf{A}_i^{k_i'}\mathsf{A}_j^{k_j'}\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\alpha(r^2\widehat{\mathsf{D}_i\rho}^{\,i}\mathsf{D}_j^2\rho) &= \mathsf{A}_i^{k_i'}\mathsf{A}_j^{k_j'}\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\alpha(r^2(\partial_i\rho)\partial_j^2\rho)\\ &+ Ch_i^2\int_{-1}^{1}(1-|\sigma|)^2\mathsf{A}_i^{k_i'}\mathsf{A}_j^{k_j'}\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\alpha(r^2\partial_i^3\rho(.+\sigma\boldsymbol{h}_i)\partial_j^2\rho)\,d\sigma\\ &+ Ch_j^2\int_{-1}^{1}(1-|\sigma|)^3\;\mathsf{A}_i^{k_i'}\mathsf{A}_j^{k_j'}\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\alpha(r^2(\partial_i\rho)\partial_j^4\rho(.+\sigma\boldsymbol{h}_j)\,d\sigma\\ &+ Ch_i^2h_j^2\iint_{[-1,1]^2}(1-|\sigma|)^2(1-|\sigma'|)^3\\ &\qquad\times\mathsf{A}_i^{k_i'}\mathsf{A}_j^{k_j'}\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\alpha(r^2\partial_i^3\rho(.+\sigma\boldsymbol{h}_i)\partial_j^4\rho(.+\sigma'\boldsymbol{h}_j))\,d\sigma\,d\sigma'\\ &= \partial_i^{k_i}\partial_j^{k_j}\partial^\alpha(r^2(\partial_i\rho)\partial_j^2\rho) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2).\end{aligned}$$

The last result follows similarly. ∎

Appendix C. Proofs of intermediate results in Section 3.

C.1. Proof of Lemma 3.3. From the forms of $A_1 v$ and $B_1 v$ we have $I_{11} = \sum_{k,l\in\{t,1,\dots,d\}} \mathscr{Q}_{kl}$ with

$$\begin{aligned}
\mathscr{Q}_{tt} &= 2\operatorname{Re}\iiint_Q r(\partial_t\rho)\,(\partial_t^2 v)\partial_t v^*\,dt,\\
\mathscr{Q}_{ti} &= 2\operatorname{Re}\iiint_Q \xi_{1,i}\xi_{2,i} r\overline{D_i\rho}^{\,i}\,(\partial_t^2 v)\overline{D_i v^*}^{\,i}\,dt, \quad i\in[\![1,d]\!],\\
\mathscr{Q}_{it} &= 2\operatorname{Re}\iiint_Q \xi_{1,i} r^2(\partial_t\rho)\overline{\widetilde{\rho}}^{\,i}\,\bar D_i(\xi_{2,i}D_i v)\partial_t v^*\,dt, \quad i\in[\![1,d]\!],\\
\mathscr{Q}_{ii} &= 2\operatorname{Re}\iint_Q \xi_{1,i}^2\xi_{2,i} r^2\overline{\widetilde{\rho}}^{\,i}\,\overline{D_i\rho}^{\,i}\,\bar D_i(\xi_{2,i}D_i v)\overline{D_i v^*}^{\,i}\,dt, \quad i\in[\![1,d]\!],\\
\mathscr{Q}_{ij} &= 2\operatorname{Re}\iiint_Q \xi_{1,i}\xi_{1,j}\xi_{2,j} r^2\overline{\widetilde{\rho}}^{\,i}\,\overline{D_j\rho}^{\,j}\,\bar D_i(\xi_{2,i}D_i v)\overline{D_j v^*}^{\,j}\,dt, \quad i,j\in[\![1,d]\!],\ i\neq j.
\end{aligned}$$

We start by computing each term.

Computation of $\mathscr{Q}_{tt}$. We set $q_{tt} = -\partial_t\big(r(\partial_t\rho)\big)$. An integration by parts w.r.t. t yields

$$\mathscr{Q}_{tt} = \iiint_Q q_{tt}\,|\partial_t v|^2\,dt - s\lambda\left[\iint_\Omega \varphi(\partial_t\psi)\,|\partial_t v|^2\right]_0^{\mathcal{T}}.$$

Lemma C.1. *We have*

$$q_{tt} = s\lambda^2\varphi(\partial_t\psi)^2 + s\lambda\varphi\mathcal{O}(1).$$

The estimation of follows from Lemma 2.7.

Computation of $\mathscr{Q}_{ti}$. Setting $p_{ti} = -\xi_{1,i}\xi_{2,i} r\overline{D_i\rho}^{\,i}$ and $q_{ti} = \partial_t p_{ti}$ we have, by integration by parts w.r.t. t since $v_{|t=0} = 0$,

$$\begin{aligned}
\mathscr{Q}_{ti} &= 2\operatorname{Re}\iiint_Q (\partial_t v)\partial_t\big(p_{ti}\overline{D_i v^*}^{\,i}\big)\,dt - 2\operatorname{Re}\iint_\Omega \big(p_{ti}(\partial_t v)\overline{D_i v^*}^{\,i}\big)(\mathcal{T})\\
&= 2\operatorname{Re}\iiint_Q q_{ti}(\partial_t v)\overline{D_i v^*}^{\,i}\,dt + \underbrace{2\operatorname{Re}\iiint_Q p_{ti}(\partial_t v)\partial_t\overline{D_i v^*}^{\,i}\,dt}_{\mathscr{Q}_{ti}^a},
\end{aligned}$$

using that $p_{ti}(\mathcal{T}) = 0$ for $\psi_{|t=\mathcal{T}} = \text{Cst}$. As $v_{|\partial\Omega} = 0$ with Proposition 2.4, Lemma 2.2, and a discrete integration by parts w.r.t. x_i, we then write

$$\begin{aligned}
\mathscr{Q}_{ti}^a &= 2\operatorname{Re}\iiint_Q \widetilde{p_{ti}(\partial_t v)}^{\,i}\,\partial_t D_i v^*\,dt = 2\operatorname{Re}\iiint_Q \widetilde{p_{ti}}^{\,i}\widetilde{\partial_t v}^{\,i}\,\partial_t D_i v^*\,dt + \frac{h_i^2}{2}\iiint_Q (D_i p_{ti})|\partial_t D_i v|^2\,dt\\
&= -\iiint_Q (\bar D_i\widetilde{p_{ti}}^{\,i})|\partial_t v|^2\,dt + \frac{h_i^2}{2}\iiint_Q (D_i p_{ti})|\partial_t D_i v|^2\,dt.
\end{aligned}$$

We thus have

$$\mathscr{Q}_{ti} = -\iiint_Q (\bar D_i\widetilde{p_{ti}}^{\,i})|\partial_t v|^2\,dt + 2\operatorname{Re}\iiint_Q q_{ti}(\partial_t v)\overline{D_i v^*}^{\,i}\,dt + \frac{h_i^2}{2}\iiint_Q (D_i p_{ti})|\partial_t D_i v|^2\,dt. \tag{C.1}$$

Lemma C.2. *We have*

$$
\begin{aligned}
D_i p_{ti} &= s\lambda^2 \xi_{1,i}\xi_{2,i}\varphi(\partial_i\psi)^2 + s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
\bar{D}_i\widetilde{p_{ti}}^{i} &= s\lambda^2 \xi_{1,i}\xi_{2,i}\varphi(\partial_i\psi)^2 + s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
q_{ti} &= s\lambda^2 \xi_{1,i}\xi_{2,i}\varphi(\partial_t\psi)(\partial_i\psi) + s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2).
\end{aligned}
$$

Proof. We set $\alpha = -\xi_{1,i}\xi_{2,i}$. Then $D_i p_{ti} = (D_i\alpha)\widetilde{r\overline{D_i\rho}^{i}}^{i} + \tilde{\alpha}^i D_i\big(r\overline{D_i\rho}^{i}\big)$. With Proposition 2.12 we find

$$
D_i p_{ti} = (D_i\alpha) r\partial_i\rho + \tilde{\alpha}^i(\partial_i(r\partial_i\rho)) + s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2). \tag{C.2}
$$

Then with Lemma 2.7 we obtain the estimate of $D_i p_{ti}$ as $D_i\alpha = \mathcal{O}(1)$. Averaging (C.2) we obtain

$$
\begin{aligned}
\overline{D_i p_{ti}}^{i} &= \overline{D_i\alpha}^{i}\,\overline{r\partial_i\rho}^{i} + \overline{\tilde{\alpha}^i}^{i}\,\overline{\partial_i(r\partial_i\rho)}^{i} + \frac{h^2}{4}(\bar{D}_i D_i\alpha)(\bar{D}_i(r\partial_i\rho))\\
&\quad + \frac{h^2}{4}(\bar{D}_i\tilde{\alpha}^i)\bar{D}_i(\partial_i(r\partial_i\rho)) + s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2).
\end{aligned}
$$

By Lemma 2.10 we have

$$
\overline{D_i\alpha}^{i}\,\overline{r\partial_i\rho}^{i} = s\lambda\varphi\mathcal{O}(1) + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s). \tag{C.3}
$$

as $\overline{D_i\alpha}^{i} = \mathcal{O}(1)$. Note also that $\overline{\tilde{\alpha}^i}^{i} = \alpha + h\mathcal{O}(1)$. Then by Lemma 2.10 and Lemma 2.7 we have

$$
\overline{\tilde{\alpha}^i}^{i}\,\overline{\partial_i(r\partial_i\rho)}^{i} = -\alpha s\lambda^2\varphi(\partial_i\psi)^2 + s\lambda\varphi\mathcal{O}(1) + \mathcal{O}_{\lambda,\mathfrak{K}}(sh). \tag{C.4}
$$

Since $h\bar{D}_i D_i\alpha = \mathcal{O}(1)$, by Lemma 2.10 we obtain

$$
\frac{h^2}{4}(\bar{D}_i D_i\alpha)(\bar{D}_i(r\partial_i\rho)) = \mathcal{O}_{\lambda,\mathfrak{K}}(sh). \tag{C.5}
$$

Similarly we have

$$
\frac{h^2}{4}(\bar{D}_i\tilde{\alpha}^i)\bar{D}_i(\partial_i(r\partial_i\rho)) = h\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \tag{C.6}
$$

as $\bar{D}_i\tilde{\alpha}^i = \overline{D_i\alpha}^{i} = \mathcal{O}(1)$. Collecting estimates (C.3)–(C.6), we obtain the second result.

Finally we write $q_{ti} = \alpha\partial_t(r\overline{D_i\rho}^{i})$; Proposition 2.12 and Lemma 2.7 yield the estimates for q_{ti}. □

Computation of $\mathscr{Q}_{it}$. We set $p_{it} = -\xi_{1,i}r^2(\partial_t\rho)\overline{\tilde{\rho}^i}^{i}$ and $q_{it} = \xi_{2,i}D_i p_{it}$. Since $v_{|\partial\Omega} = 0$, with a discrete integration by parts w.r.t. x_i (Proposition 2.4) we then write

$$
\begin{aligned}
\mathscr{Q}_{it} &= 2\operatorname{Re}\iiint_Q D_i(p_{it}\partial_t v^*)\,\xi_{2,i}D_i v\,dt = 2\operatorname{Re}\iiint_Q \Big(q_{it}\,\widetilde{\partial_t v^*}^{i} + \xi_{2,i}\widetilde{p_{it}}^{i}\,(\partial_t D_i v^*)\Big) D_i v\,dt\\
&= 2\operatorname{Re}\iiint_Q \overline{q_{it}\,D_i v}^{i}\,\partial_t v^*\,dt - \iiint_Q \xi_{2,i}(\partial_t\widetilde{p_{it}}^{i})|D_i v|^2\,dt + \iint_\Omega \xi_{2,i}\big(\widetilde{p_{it}}^{i}|D_i v|^2\big)(T),
\end{aligned}
$$

after an integration by parts w.r.t. t, to yield

$$\begin{aligned}\mathscr{Q}_{it} &= 2\,\mathrm{Re} \iiint_Q \overline{q_{it}}^i\, \overline{D_i v}^i \partial_t v^*\, dt + \frac{h_i^2}{2}\,\mathrm{Re} \iiint_Q \bar{D}_i(q_{it})(\bar{D}_i D_i v)\partial_t v^*\, dt \\ &\quad - \iiint_Q \xi_{2,i}(\partial_t \widetilde{p_{it}}^i)|D_i v|^2\, dt + \iint_\Omega \xi_{2,i}\big(\widetilde{p_{it}}^i |D_i v|^2\big)(T).\end{aligned}$$

Lemma C.3. *We have*

$$\begin{aligned}\xi_{2,i}\widetilde{p_{it}}^i &= s\lambda \xi_{1,i}\xi_{2,i}\varphi(\partial_t\psi) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\ \xi_{2,i}\partial_t\widetilde{p_{it}}^i &= s\lambda^2 \xi_{1,i}\xi_{2,i}\varphi(\partial_t\psi)^2 + s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\ \overline{q_{it}}^i &= s\lambda^2\xi_{1,i}\xi_{2,i}\varphi(\partial_i\psi)(\partial_t\psi) + s\lambda\xi_{1,i}\xi_{2,i}\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\ h\bar{D}_i(q_{it}) &= s\lambda\varphi\mathcal{O}(1) + \mathcal{O}_{\lambda,\mathfrak{K}}(sh).\end{aligned}$$

Proof. The first three estimates follow from Proposition 2.13 and Corollary 2.8 following the method of the proof of Lemma C.2 (see also the proof of similar technical lemmata in [BHL09a, Appendix B]).

For the fourth estimate we first write

$$\begin{aligned}h_i\bar{D}_i q_{it} &= h_i\bar{D}_i(\xi_{2,i}D_i p_{it}) = h_i(\bar{D}_i\xi_{2,i})\overline{D_i p_{it}}^i + h_i\overline{\xi_{2,i}}^i\bar{D}_i D_i p_{it}\\ &= \mathcal{O}_{\lambda,\mathfrak{K}}(sh) + h_i\mathcal{O}(1)\bar{D}_i D_i p_{it},\end{aligned}$$

following the method of the proof of Lemma C.2. We then write

$$D_i p_{it} = -(D_i\xi_{1,i})\widetilde{r^2(\partial_t\rho)\overline{\tilde{\rho}}^i}^i - \widetilde{\xi_{1,i}}^i D_i(r^2(\partial_t\rho)\overline{\tilde{\rho}}^i).$$

and obtain

$$\begin{aligned}h_i\bar{D}_i D_i p_{it} &= -h_i(\bar{D}_i D_i\xi_{1,i})\overline{\widetilde{r^2(\partial_t\rho)\overline{\tilde{\rho}}^i}^i}^i - 2h_i\overline{D_i\xi_{1,i}}^i\,\overline{D_i(r^2(\partial_t\rho)\overline{\tilde{\rho}}^i)}^i\\ &\quad - h_i\overline{\widetilde{\xi_{1,i}}^i}^i\bar{D}_i D_i(r^2(\partial_t\rho)\overline{\tilde{\rho}}^i)\\ &= s\lambda\varphi\mathcal{O}(1) + \mathcal{O}_{\lambda,\mathfrak{K}}(sh),\end{aligned}$$

arguing as in the proof of Lemma C.2, as $D_i\xi_{1,i} = \mathcal{O}(1)$. The result follows. ∎

Computation of $\mathscr{Q}_{ii}$. We set $p_{ii} = -\xi_{1,i}^2\xi_{2,i}r^2\overline{\tilde{\rho}}^i\,\overline{D_i\rho}^i$ and $q_{ii} = D_i(\overline{\xi_{2,i}}^i p_{ii})$. By Lemmata 2.1 and 2.4, we have

$$\begin{aligned}\mathscr{Q}_{ii} &= \iiint_Q q_{ii}\,|D_i v|^2\, dt - \iint_{Q_i}\Big(\big(\overline{\xi_{2,i}}^i p_{ii}\big)_{N_i+1}|D_i v|^2_{N_i+\frac12} - \big(\overline{\xi_{2,i}}^i p_{ii}\big)_0|D_i v|^2_{\frac12}\Big)\, dt\\ &\quad - 2\iiint_Q p_{ii}\bar{D}_i(\xi_{2,i})\big|\overline{D_i v}^i\big|^2\, dt.\end{aligned}$$

For the first term we write

$$\begin{aligned}\iiint_Q q_{ii}\,|D_i v|^2\, dt &= -\iiint_Q q_{ii}\,|D_i v|^2\, dt + \underbrace{2\iiint_Q \overline{q_{ii}\,|D_i v|^2}^i\, dt}_{=\mathscr{Q}_{ii}^a}\\ &\quad + h_i\iint_{Q_i}\big((q_{ii})_{\frac12}|D_i v|^2_{\frac12} + (q_{ii})_{N_i+\frac12}|D_i v|^2_{N_i+\frac12}\big)\end{aligned}$$

by Proposition 2.4 and with Lemma 2.2 we further have

$$\mathscr{Q}_{ii}^{a} = 2 \iiint\limits_{Q} \overline{q_{ii}}^{i} \, \overline{|D_i v|^2}^{i} \, dt + \frac{h_i^2}{2} \iiint\limits_{Q} (\bar{D}_i q_{ii}) \, \bar{D}_i |D_i v|^2 \, dt.$$

A further use of Lemma 2.2 and a discrete integration by parts w.r.t. x_i (Proposition 2.4) yield,

$$\begin{aligned}\mathscr{Q}_{ii}^{a} &= 2 \iiint\limits_{Q} \overline{q_{ii}}^{i} \left| \overline{D_i v}^{i} \right|^2 dt + \frac{h_i^2}{2} \iiint\limits_{Q} \overline{q_{ii}}^{i} |\bar{D}_i D_i v|^2 \, dt - \frac{h_i^2}{2} \iiint\limits_{Q} (D_i \bar{D}_i q_{ii}) |D_i v|^2 \, dt \\ &\quad + \frac{h_i^2}{2} \iint\limits_{Q_i} \Big((\bar{D}_i q_{ii})_{N_i+1} |D_i v|^2_{N_i+\frac{1}{2}} - (\bar{D}_i q_{ii})_0 |D_i v|^2_{\frac{1}{2}} \Big) \, dt.\end{aligned}$$

We thus have

$$\begin{aligned}\mathscr{Q}_{ii} &= - \iiint\limits_{Q} q_{ii} \, |D_i v|^2 \, dt + 2 \iiint\limits_{Q} \overline{q_{ii}}^{i} \left| \overline{D_i v}^{i} \right|^2 dt - 2 \iiint\limits_{Q} p_{ii} \bar{D}_i(\xi_{2,i}) \left| \overline{D_i v}^{i} \right|^2 dt \\ &\quad - \iint\limits_{Q_i} \Big(\big(\overline{\xi_{2,i}}^{i} p_{ii}\big)_{N_i+1} |D_i v|^2_{N_x+\frac{1}{2}} - \big(\overline{\xi_{2,i}}^{i} p_{ii}\big)_0 |D_i v|^2_{\frac{1}{2}} \Big) \, dt \\ &\quad + h_i \iint\limits_{Q_i} \Big((\overline{q_{ii}}^{i})_{N_i+1} |D_i v|^2_{N_x+\frac{1}{2}} + (\overline{q_{ii}}^{i})_0 |D_i v|^2_{\frac{1}{2}} \Big) \, dt \\ &\quad + \frac{h_i^2}{2} \iiint\limits_{Q} \overline{q_{ii}}^{i} |\bar{D}_i D_i v|^2 \, dt - \frac{h_i^2}{2} \iiint\limits_{Q} (D_i \bar{D}_i q_{ii}) |D_i v|^2 \, dt.\end{aligned}$$

LEMMA C.4. *We have*

$$\begin{aligned}\overline{\xi_{2,i}}^{i} p_{ii} &= -\big(\xi_{1,i}^2 \xi_{2,i}^2 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big) \, r \overline{D_i \rho}^{i}, \\ p_{ii} \bar{D}_i(\xi_{2,i}) &= s\lambda\varphi \mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\ q_{ii} &= s\lambda^2 \xi_{1,i}^2 \xi_{2,i}^2 \varphi (\partial_i \psi)^2 + s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\ \overline{q_{ii}}^{i} &= s\lambda^2 \xi_{1,i}^2 \xi_{2,i}^2 \varphi (\partial_i \psi)^2 + s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\ h_i^2 D_i \bar{D}_i q_{ii} &= s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh).\end{aligned}$$

Moreover for h_i sufficiently small we have

$$\begin{aligned}(\overline{q_{ii}}^{i})_{N_i+1} &\geq s\lambda(\varphi)_{N_i+\frac{1}{2}} \mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\ (\overline{q_{ii}}^{i})_0 &\geq s\lambda(\varphi)_{\frac{1}{2}} \mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh).\end{aligned} \tag{C.7}$$

Proof. The first estimate follows from Proposition 2.9. The next three estimates all follow from Proposition 2.13 and Corollary 2.8, following the method of the proof of Lemma C.2.

To estimate $h_i^2 D_i \bar{D}_i q_{ii}$, introducing $\alpha = -\xi_{1,i}^2 \xi_{2,i} \overline{\xi_{2,i}}^{i}$ and $\gamma = r^2 \overline{\tilde{\rho}^{i}}^{i} \, \overline{D_i \rho}^{i}$ we first write

$$D_i \bar{D}_i q_{ii} = (D_i \bar{D}_i D_i \alpha) \widetilde{\overline{\tilde{\gamma}^{i}}^{i}}^{i} + 3 \widetilde{\bar{D}_i D_i \alpha}^{i} \, \widetilde{\overline{D_i \gamma}^{i}}^{i} + 3 \widetilde{\overline{D_i \alpha}^{i}}^{i} \, \widetilde{\bar{D}_i D_i \gamma}^{i} + \widetilde{\overline{\tilde{\alpha}^{i}}^{i}}^{i} (D_i \bar{D}_i D_i \gamma).$$

We note that we have

$$h^2 D_i \bar{D}_i D_i \alpha = \mathcal{O}(1), \quad h \widetilde{\bar{D}_i D_i \alpha}^{i} = \mathcal{O}(1), \quad \widetilde{\overline{D_i \alpha}^{i}}^{i} = \mathcal{O}(1), \quad \widetilde{\overline{\tilde{\alpha}^{i}}^{i}}^{i} = \mathcal{O}(1),$$

and, with Proposition 2.13,

$$\widetilde{\widetilde{\gamma}^{i}}^{i} = s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2), \quad \widetilde{\overline{D_i\gamma}^{i}}^{i} = s\mathcal{O}_{\lambda,\mathfrak{K}}(1), \quad \widetilde{\bar{D}_iD_i\gamma}^{i} = s\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$
$$hD_i\bar{D}_iD_i\gamma = s\mathcal{O}_{\lambda,\mathfrak{K}}(1).$$

The estimate for $h_i^2 D_i\bar{D}_i q_{ii}$ then follows.

For the second part of the proof we only address the first inequality in (C.7). The second inequality follows similarly. We have

$$\overline{q_{ii}}^{i} = \underbrace{\overline{-D_i\big(\xi_{1,i}^2\xi_{2,i}\overline{\xi_{2,i}}^{i}\big)r^2\widetilde{\overline{\widetilde{\rho}}^{i}\,\overline{D_i\rho}^{i}}^{i}}^{i}}_{=s\lambda\varphi\mathcal{O}(1)+s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)} + \overline{\underbrace{\widetilde{\xi_{1,i}^2\xi_{2,i}\overline{\xi_{2,i}}^{i}}^{i}}_{\geq 0}\, b_{ii}}^{i}$$

where the estimation of the first term follows as in the proof of Lemma C.2 and with $b_{ii} = D_i(-r^2\overline{\widetilde{\rho}}^{i}\,\overline{D_i\rho}^{i})$.

It remains thus to prove that $(b_{ii})_{k_i} \geq 0$, $k_i = N_i + \frac{1}{2}, N_i + \frac{3}{2}$, for h_i sufficiently small. Observing that $\partial_i^2\varphi(x) = \lambda^2(\partial_i\psi)^2\varphi + \lambda(\partial_i^2\psi)\varphi$, with the assumption made on ψ in the neighborhood of the boundary $\partial_i\Omega$, we see that the function $x_i \mapsto \varphi(t, x_1, \dots, x_d)$ is convex in a neighborhood of $\{x_i = L_i\}$. It thus follows that

$$\varphi_{k_i+1} + \varphi_{k_i-1} - 2\varphi_{k_i} \geq 0,$$

for k_ih_i close to $L_i = (N_i+1)h_i$. As $\rho = e^{-s\varphi}$ it follows that

$$\frac{\rho_{k_i+1}}{\rho_{k_i}} \leq \frac{\rho_{k_i}}{\rho_{k_i-1}}, \quad \text{for } k_ih_i \text{ close to } (N_i+1)h_i. \tag{C.8}$$

We now write

$$(-r^2\overline{\widetilde{\rho}}^{i}\,\overline{D_i\rho}^{i})_{k_i} = \frac{1}{8h_i}\Big(\big(1+\frac{\rho_{k_i-1}}{\rho_{k_i}}\big)^2 - \big(1+\frac{\rho_{k_i+1}}{\rho_{k_i}}\big)^2\Big),$$

which gives

$$h_i(b_{ii})_{k_i+\frac{1}{2}} = \frac{1}{8h_i}\Big(\underbrace{\big(1+\frac{\rho_{k_i}}{\rho_{k_i+1}}\big)^2 - \big(1+\frac{\rho_{k_i-1}}{\rho_{k_i}}\big)^2}_{\geq 0} + \underbrace{\big(1+\frac{\rho_{k_i+1}}{\rho_{k_i}}\big)^2 - \big(1+\frac{\rho_{k_i+2}}{\rho_{k_i+1}}\big)^2}_{\geq 0}\Big),$$

by (C.8) if k_ih_i close to $L_i = (N_i+1)h_i$. Inequality (C.7) thus follows for h_i small, noting that $(\varphi)_{k_i+1} = (\varphi)_{k_i+\frac{1}{2}} + h^2\mathcal{O}_\lambda(1)$. ◻

Computation of $\mathscr{Q}_{ij}$, $i \neq j$. We set $p_{ij} = -\xi_{1,i}\xi_{1,j}\xi_{2,j}r^2\overline{\widetilde{\rho}}^{i}\,\overline{D_j\rho}^{j}$ and $q_{ij} = \xi_{2,i}D_ip_{ij}$. As $v_{|\partial\Omega} = 0$, a discrete integration by parts w.r.t. x_i (see Lemma 2.4) yields

$$\mathscr{Q}_{ij} = 2\,\mathrm{Re}\iiint_Q \xi_{2,i}D_i\big(p_{ij}\,\overline{D_jv^*}^{j}\big)\,D_iv\,dt,$$

which can be written as $\mathscr{Q}_{ij} = \mathscr{Q}_{ij}^a + \mathscr{Q}_{ij}^b$ with

$$\mathscr{Q}_{ij}^a = 2\,\mathrm{Re}\iiint_Q q_{ij}\,\widetilde{\overline{D_jv^*}^{j}}^{i}\,D_iv\,dt, \quad \mathscr{Q}_{ij}^b = 2\,\mathrm{Re}\iiint_Q \xi_{2,i}\widetilde{p_{ij}}^{i}\,\overline{D_jD_iv^*}^{j}\,D_iv\,dt.$$

By Proposition 2.4 we write

$$
\begin{aligned}
\mathscr{Q}^a_{ij} &= 2\operatorname{Re} \iiint_Q \overline{q_{ij}\, D_i v}^{\,i}\; \overline{D_j v^*}^{\,j}\, dt \\
&= 2\operatorname{Re} \iiint_Q \overline{q_{ij}}^{\,i}\; \overline{D_i v}^{\,i}\overline{D_j v^*}^{\,j}\, dt + \frac{h_i^2}{2}\operatorname{Re}\iiint_Q (\bar{D}_i q_{ij})\,(\bar{D}_i D_i v)\,\overline{D_j v^*}^{\,j}\, dt.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\mathscr{Q}^b_{ij} &= 2\operatorname{Re} \iiint_Q \widetilde{\xi_{2,i}\widetilde{p_{ij}}^{\,i}\, D_i v}^{\,j}\; D_j D_i v^*\, dt \\
&= 2\operatorname{Re} \iiint_Q \widetilde{\xi_{2,i}\widetilde{p_{ij}}^{\,i}}^{\,j}\; \widetilde{D_i v}^{\,j}\; D_j D_i v^*\, dt + \frac{h_j^2}{2}\iiint_Q D_j(\xi_{2,i}\widetilde{p_{ij}}^{\,i})\,|D_j D_i v|^2\, dt \\
&= -\iiint_Q \overline{D_j(\xi_{2,i}\widetilde{p_{ij}}^{\,i})}^{\,j}\; |D_i v|^2\, dt + \frac{h_j^2}{2}\iiint_Q D_j(\xi_{2,i}\widetilde{p_{ij}}^{\,i})\,|D_j D_i v|^2\, dt.
\end{aligned}
$$

We thus have

$$
\begin{aligned}
\mathscr{Q}_{ij} = &-\iiint_Q \overline{D_j(\xi_{2,i}\widetilde{p_{ij}}^{\,i})}^{\,j}\; |D_i v|^2\, dt + 2\operatorname{Re}\iiint_Q \overline{q_{ij}}^{\,i}\; \overline{D_i v}^{\,i}\overline{D_j v^*}^{\,j}\, dt \\
&+ \frac{h_i^2}{2}\operatorname{Re}\iiint_Q (\bar{D}_i q_{ij})\,(\bar{D}_i D_i v)\,\overline{D_j v^*}^{\,j} dt + \frac{h_j^2}{2}\iiint_Q D_j(\xi_{2,i}\widetilde{p_{ij}}^{\,i})\,|D_j D_i v|^2\, dt
\end{aligned}
\tag{C.9}
$$

Lemma C.5. *We have*

$$
\begin{aligned}
\overline{D_j(\xi_{2,i}\widetilde{p_{ij}}^{\,i})}^{\,j} &= s\lambda^2 \xi_{1,i}\xi_{2,i}\xi_{1,j}\xi_{2,j}\varphi(\partial_j\psi)^2 + s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
\overline{q_{ij}}^{\,i} &= s\lambda^2 \xi_{1,i}\xi_{2,i}\xi_{1,j}\xi_{2,j}\varphi(\partial_i\psi)(\partial_j\psi) + s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
D_j(\xi_{2,i}\widetilde{p_{ij}}^{\,i}) &= s\lambda^2 \xi_{1,i}\xi_{2,i}\xi_{1,j}\xi_{2,j}\varphi(\partial_j\psi)^2 + s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
h\bar{D}_i q_{ij} &= s\lambda\varphi\mathcal{O}(1) + \mathcal{O}_{\lambda,\mathfrak{K}}(sh).
\end{aligned}
$$

The estimates all follow from Proposition 2.13 and Corollary 2.8, arguing as in the proof of Lemma C.2.

Estimate of I_{11}. We now collect the different terms that we have just computed and use Lemmata C.1 to C.5 to write

$$
I_{11} = I'_{11} + Y_{11} + I''_{11} + I'''_{11} - (J_{11} + Z_{11} + Z'_{11} + Z''_{11}),
$$

where

$$
\begin{aligned}
I'_{11} = &-s\lambda^2 \iiint_Q \varphi|\nabla_\xi\psi|^2|\partial_t v|^2\, dt - s\lambda^2 \sum_{i\in[\![1,d]\!]}\iiint_Q \varphi\xi_{1,i}\xi_{2,i}|\nabla_\xi\psi|^2|D_i v|^2\, dt \\
&+ s\lambda \sum_{i\in[\![1,d]\!]}\iint_\Omega \Big(\varphi\xi_{1,i}\xi_{2,i}(\partial_t\psi)|D_i v|^2\Big)(\mathcal{T}) - s\lambda\Big[\iint_\Omega \varphi(\partial_t\psi)|\partial_t v|^2\Big]_0^{\mathcal{T}}
\end{aligned}
$$

and

$$
\begin{aligned}
Y_{11} = \sum_{i\in[\![1,d]\!]}\iint_{Q_i} \Big(&\big((\xi_{1,i}^2\xi_{2,i}^2 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh))\, r\overline{D_i\rho}^{\,i}\big)_{N_i+1}|D_i v|^2_{N_i+\frac12} \\
&- \big((\xi_{1,i}^2\xi_{2,i}^2 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh))\, r\overline{D_i\rho}^{\,i}\big)_0|D_i v|^2_{\frac12}\Big)\, dt,
\end{aligned}
$$

and

$$
\begin{aligned}
I''_{11} &= 2s\lambda^2 \iiint_Q \varphi\Big((\partial_t\psi)^2|\partial_t v|^2 + \sum_{i\in[\![1,d]\!]} \xi_{1,i}^2\xi_{2,i}^2(\partial_i\psi)^2\big|\overline{D_i v}^i\big|^2\Big)\,dt \\
&\quad + 2s\lambda^2 \operatorname{Re}\iiint_Q \varphi\Big(2(\partial_t\psi)\partial_t v \sum_{i\in[\![1,d]\!]} \xi_{1,i}\xi_{2,i}(\partial_i\psi)\overline{D_i v^*}^i \\
&\qquad + \sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \xi_{1,i}\xi_{2,i}\xi_{1,j}\xi_{2,j}(\partial_i\psi)(\partial_j\psi)\overline{D_i v}^i\overline{D_j v^*}^j\Big)\,dt, \\
&= 2s\lambda^2 \iiint_Q \varphi\Big|(\partial_t\psi)\partial_t v + \sum_{i\in[\![1,d]\!]} \xi_{1,i}\xi_{2,i}(\partial_i\psi)\overline{D_i v}^i\Big|^2\,dt \\
&\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
I'''_{11} &= \sum_{i\in[\![1,d]\!]} \frac{s\lambda^2 h_i^2}{2} \iiint_Q \varphi\xi_{1,i}\xi_{2,i}(\partial_i\psi)^2|D_i\partial_t v|^2\,dt \\
&\quad + \sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \frac{s\lambda^2 h_j^2}{2} \iiint_Q \varphi\xi_{1,i}\xi_{2,i}\xi_{1,j}\xi_{2,j}(\partial_j\psi)^2|D_iD_j v|^2\,dt \\
&\quad + \sum_{i\in[\![1,d]\!]} \frac{s\lambda^2 h_i^2}{2} \iiint_Q \varphi\xi_{1,i}^2\xi_{2,i}^2(\partial_i\psi)^2|\bar{D}_iD_i v|^2\,dt \\
&\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
J_{11} &= \sum_{i\in[\![1,d]\!]} \iint_\Omega \delta_{11,i}|D_i v|^2(\mathcal{T}) \\
&\quad + \sum_{i\in[\![1,d]\!]} \iint_{Q_i} \Big((\delta_{11,i}^{(2)})_{N_x+\frac12}|D_i v|^2_{N_x+\frac12} + (\delta_{11,i}^{(2)})_{\frac12}|D_i v|^2_{\frac12}\Big)\,dt
\end{aligned}
$$

with $\delta_{11,i} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$, and $\delta_{11,i}^{(2)} = h(s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh))$, and

$$
Z_{11} = \iiint_Q \beta'_{11}|\partial_t v|^2\,dt + \sum_{i\in[\![1,d]\!]} \iiint_Q \nu'_{11,i}|D_i v|^2\,dt + \sum_{i\in[\![1,d]\!]} \iiint_Q \overline{\nu}'_{11,i}\big|\overline{D_i v}^i\big|^2\,dt
$$

and

$$
Z'_{11} = \operatorname{Re}\sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \iiint_Q \alpha_{11,ij}\,\overline{D_i v}^i\overline{D_j v^*}^j\,dt + \operatorname{Re}\sum_{i\in[\![1,d]\!]} \iiint_Q \alpha_{11,ti}\,\overline{D_i v}^i\,\partial_t v^*\,dt
$$

where β'_{11}, $\nu'_{11,i}$, $\overline{\nu}'_{11,i}$, $\alpha_{11,ij}$, and $\alpha_{11,ti}$ are of the form $s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$, and

$$
\begin{aligned}
Z''_{11} = &\sum_{i\in[\![1,d]\!]} \iiint_Q \gamma''_{11,ti}|D_i\partial_t v|^2\,dt + \sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \iiint_Q \gamma''_{11,ij}|D_iD_j v|^2\,dt + \sum_{i\in[\![1,d]\!]} \iiint_Q \gamma''_{11,ii}|\bar{D}_iD_i v|^2\,dt \\
&+ \operatorname{Re}\sum_{i\in[\![1,d]\!]} \iiint_Q \gamma''_{11,iit}(\bar{D}_iD_i v)\partial_t v^*\,dt + \operatorname{Re}\sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \iiint_Q \gamma''_{11,iij}(\bar{D}_iD_i v)\overline{D_j v^*}^j\,dt,
\end{aligned}
$$

where $\gamma''_{11,ti}$, $\gamma''_{11,ij}$, and $\gamma''_{11,ii}$ are of the form $h^2\big(s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)$, and $\gamma''_{11,iit}$, $\gamma''_{11,iij}$ are of the form $h\big(s\lambda\varphi\mathcal{O}(1) + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)$.

We conclude with Cauchy-Schwarz inequalities that yields

$$|Z'_{11}| \leq \sum_{i\in[\![1,d]\!]} \iiint_Q \alpha_{11,i} \left|\overline{D_i v}^i\right|^2 dt + \iiint_Q \alpha_{11,t} \left|\partial_t v\right|^2 dt,$$

with $\alpha_{11,i}$ and $\alpha_{11,t}$ of the form $s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$, and

$$\begin{aligned}|Z''_{11}| \leq &\sum_{i\in[\![1,d]\!]} \iiint_Q \gamma'''_{11,ti}|D_i\partial_t v|^2 dt + \sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \iiint_Q \gamma'''_{11,ij}|D_iD_jv|^2 dt + \sum_{i\in[\![1,d]\!]} \iiint_Q \gamma'''_{11,ii}|\bar{D}_iD_iv|^2 dt\\ &+ \iiint_Q \gamma'''_{11,t}|\partial_t v|^2 dt + \sum_{i\in[\![1,d]\!]} \iiint_Q \gamma'''_{11,i}|\overline{D_i v}^i|^2 dt,\end{aligned}$$

with $\gamma'''_{11,ti}$, $\gamma'''_{11,ij}$, and $\gamma'''_{11,ii}$ are of the form $h^2\big(s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)$ and $\gamma'''_{11,t}$ and $\gamma'''_{11,i}$ are of the form $s\lambda\varphi\mathcal{O}(1) + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)$. ☐

C.2. Proof of Lemma 3.5. As compared to the computation of the counterpart of I_{21} in the proof of the semi-discrete Carleman estimate in [BHL09a] (also denoted I_{21} there) we need to compute the following additional terms,

$$\mathscr{Q}_{ij} = -2\,\mathrm{Re} \iiint_Q p_{ij}\, \overline{\tilde{v}^j}^j\, \overline{D_i v^*}^i\, dt,$$

for $i \neq j$, where $p_{ij} = -\xi_{1,i}\xi_{2,i}\xi_{1,j}\xi_{2,j} r^2(\bar{D}_jD_j\rho)\,\overline{D_i\rho}^i$.

With Proposition 2.4, we have

$$\begin{aligned}\mathscr{Q}_{ij} &= -2\,\mathrm{Re} \iiint_Q \widetilde{p_{ij}\overline{D_i v^*}^i}^j\, \tilde{v}^j\, dt\\ &= \underbrace{-2\,\mathrm{Re} \iiint_Q \widetilde{p_{ij}}^j\,\widetilde{\overline{D_i v^*}^i}^j\, \tilde{v}^j\, dt}_{\mathscr{Q}^a_{ij}} - \frac{h^2}{2}\,\mathrm{Re} \iiint_Q (D_jp_{ij})(D_j\overline{D_i v^*}^i)\tilde{v}^j\, dt.\end{aligned}$$

We now write

$$\begin{aligned}\mathscr{Q}^a_{ij} &= -2\,\mathrm{Re} \iiint_Q \widetilde{\widetilde{p_{ij}}^j\tilde{v}^j}^i\, \widetilde{D_i v^*}^j\, dt\\ &= \underbrace{-2\,\mathrm{Re} \iiint_Q \widetilde{\widetilde{p_{ij}}^j}^i\, \widetilde{\tilde{v}^j}^i\, \widetilde{D_i v^*}^j\, dt}_{\mathscr{Q}^b_{ij}} - \frac{h^2}{2} \iiint_Q (D_i\widetilde{p_{ij}}^j)\, |D_i\tilde{v}^j|^2\, dt,\end{aligned}$$

and with a discrete integration by parts in x_i (Proposition 2.4) and Lemma 2.2 we have, as $\tilde{v}^j = 0$ on $\partial_i Q$,

$$\begin{aligned}\mathscr{Q}^b_{ij} &= -\iiint_Q \widetilde{\widetilde{p_{ij}}^j}^i\, D_i|\tilde{v}^j|^2\, dt = \iiint_Q \left(\bar{D}_i\widetilde{\widetilde{p_{ij}}^j}^i\right)|\tilde{v}^j|^2\, dt\\ &= \iiint_Q \left(\bar{D}_i\widetilde{\widetilde{p_{ij}}^j}^i\right)\widetilde{|v|^2}^j\, dt - \frac{h^2}{4} \iiint_Q \left(\bar{D}_i\widetilde{\widetilde{p_{ij}}^j}^i\right)|D_jv|^2\, dt\\ &= \iiint_Q \overline{\bar{D}_i\widetilde{\widetilde{p_{ij}}^j}^i}^j\, |v|^2\, dt - \frac{h^2}{4} \iiint_Q \left(\bar{D}_i\widetilde{\widetilde{p_{ij}}^j}^i\right)|D_jv|^2\, dt\end{aligned}$$

We thus have

$$\begin{aligned}\mathscr{Q}_{ij} &= \iiint_Q \overline{\bar{D}_i \widetilde{p_{ij}}^{j}}^{i^j} |v|^2 \, dt - \frac{h^2}{2} \iiint_Q (D_i \widetilde{p_{ij}}^{j}) \, |D_i \tilde{v}^j|^2 \, dt \\ &\quad - \frac{h^2}{4} \iiint_Q \left(\bar{D}_i \widetilde{p_{ij}}^{j^i} \right) |D_j v|^2 \, dt - \frac{h^2}{2} \operatorname{Re} \iiint_Q (D_j p_{ij})(D_j \overline{D_i v^*}^i) \, \tilde{v}^j \, dt,\end{aligned}$$

LEMMA C.6. *We have*

$$\begin{aligned}\overline{\bar{D}_i \widetilde{p_{ij}}^{j}}^{i^j} &= 3s^3\lambda^4 \xi_{1,i}\xi_{2,i}\xi_{1,j}\xi_{2,j}\varphi^3(\partial_i\psi)^2(\partial_j\psi)^2 \\ &\quad + (s\lambda\varphi)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2), \\ D_i \widetilde{p_{ij}}^{j} &= s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1), \quad \bar{D}_i \widetilde{p_{ij}}^{j^i} = s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1), \quad D_j p_{ij} = s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1).\end{aligned}$$

The estimations follow from Proposition 2.13 and Corollary 2.8 arguing as in the proof of Lemmata C.2. By Young's inequality we now note that

$$\begin{aligned}&\frac{h^2}{2} \Big| \operatorname{Re} \iiint_Q (D_i p_{ij})(D_i \overline{D_j v^*}^j) \, \tilde{v}^i \, dt \Big| \\ &\qquad \le s^3(sh) \iiint_Q \mathcal{O}_{\lambda,\mathfrak{K}}(1) |\tilde{v}^i|^2 \, dt + sh^2(sh) \iiint_Q \mathcal{O}_{\lambda,\mathfrak{K}}(1) |D_i \overline{D_j v}^j|^2 \, dt \\ &\qquad \le s^3(sh) \iiint_Q \mathcal{O}_{\lambda,\mathfrak{K}}(1) \widetilde{|v|^2}^i \, dt + sh^2(sh) \iiint_Q \mathcal{O}_{\lambda,\mathfrak{K}}(1) \overline{|D_i D_j v|^2}^j \, dt \\ &\qquad \le s^3(sh) \iiint_Q \mathcal{O}_{\lambda,\mathfrak{K}}(1) |v|^2 \, dt + sh^2(sh) \iiint_Q \mathcal{O}_{\lambda,\mathfrak{K}}(1) |D_i D_j v|^2 \, dt,\end{aligned}$$

since $|\tilde{v}^i|^2 \le \widetilde{|v|^2}^i$ and using Proposition 2.4. Proceeding similarly for the term in $|D_i \tilde{v}^j|^2 = \big|\widetilde{D_i v}^j\big|^2$ we then obtain

$$\begin{aligned}\mathscr{Q}_{ij} &\ge 3s^3\lambda^4 \iiint_Q \xi_{1,i}\xi_{2,i}\xi_{1,j}\xi_{2,j}\varphi^3(\partial_i\psi)^2(\partial_j\psi)^2 \, |v|^2 \, dt + \iiint_Q \mu \, |v|^2 \, dt + \iiint_Q \nu_i \, |D_i v|^2 \, dt \\ &\quad + \iiint_Q \nu_j |D_j v|^2 \, dt + \iiint_Q \gamma |D_i D_j v|^2 \, dt,\end{aligned} \tag{C.10}$$

with

$$\begin{aligned}&\mu = (s\lambda\varphi)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\ &\nu_i = s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2), \quad \nu_j = s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2), \quad \gamma = sh^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh).\end{aligned}$$

With the computation performed in [BHL09a] (See Lemma 4.5 and its proof in Section B.4 in [BHL09a]) we then obtain the sought estimate from below for I_{21}. ◻

C.3. Proof of Lemma 3.7. We see that

$$Y = \sum_{i \in [\![1,d]\!]} \iint_{Q_i} \left((q_i)_{N_x+1} |D_i v|^2_{N_i+\frac{1}{2}} - (q_i)_0 |D_i v|^2_{\frac{1}{2}} \right) dt$$

with $q_i = (1 + \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)) \, r\overline{D_i \rho}^i$. By (3.3) we have $Y \ge 0$ for sh sufficiently small. ◻

C.4. Proof of Lemma 3.8. We choose $i \in [\![1, d]\!]$. With Lemmata 2.5 and 2.2 and Proposition 2.4, we have

$$\iiint_Q \varphi |\partial_t v|^2 \, dt = \iiint_Q \widetilde{\varphi |\partial_t v|^2} \, dt = \iiint_Q \widetilde{\varphi}^i \, \widetilde{|\partial_t v|^2}^i \, dt + \frac{h_i^2}{4} \iiint_Q D_i(\varphi) D_i |\partial_t v|^2 \, dt$$
$$= \iiint_Q \widetilde{\varphi}^i \left|\widetilde{\partial_t v}^i\right|^2 dt + \frac{h_i^2}{4}\Big(\iiint_Q \widetilde{\varphi}^i \, |D_i \partial_t v|^2 \, dt - \iiint_Q \bar{D}_i D_i(\varphi) |\partial_t v|^2 \, dt \Big).$$

We thus have

$$\iiint_Q \varphi |\partial_t v|^2 \, dt \geq \frac{h_i^2}{4} \iiint_Q \widetilde{\varphi}^i \, |D_i \partial_t v|^2 \, dt - \frac{h_i^2}{4} \iiint_Q \bar{D}_i D_i(\varphi) |\partial_t v|^2 \, dt. \tag{C.11}$$

Similarly, for $i, j \in [\![1, d]\!]$ with $i \neq j$, we obtain

$$\iiint_Q \varphi \, |D_i v|^2 \, dt \geq \frac{h_j^2}{4} \iiint_Q \overline{\varphi}^j \, |D_j D_i v|^2 \, dt - \frac{h_j^2}{4} \iiint_Q \bar{D}_j D_j(\varphi) \, |D_i v|^2 \, dt. \tag{C.12}$$

For $i \in [\![1, d]\!]$, we also write

$$\iiint_Q \varphi \, |D_i v|^2 \, dt = \frac{h_i}{2} \iint_{Q_i} \Big((\varphi \, |D_i v|^2)_{\frac{1}{2}} + (\varphi \, |D_i v|^2)_{N_i + \frac{1}{2}} \Big) \, dt + \underbrace{\iiint_Q \overline{\varphi \, |D_i v|^2}^i \, dt}_{= \mathscr{Q}_i},$$

by Proposition 2.4, and Lemma 2.2 yields

$$\mathscr{Q}_i = \iiint_Q \overline{\varphi}^i \, \overline{|D_i v|^2}^i \, dt + \frac{h_i^2}{4} \iiint_Q (\bar{D}_i \varphi) \, \bar{D}_i |D_i v|^2 \, dt$$
$$= \iiint_Q \overline{\varphi}^i \, \left|\overline{D_i v}^i\right|^2 dt + \frac{h_i^2}{4} \iiint_Q \overline{\varphi}^i \, \left|\bar{D}_i D_i v\right|^2 dt - \frac{h_i^2}{4} \iiint_Q (D_i \bar{D}_i \varphi) \, |D_i v|^2 \, dt$$
$$+ \frac{h_i^2}{4} \iint_{Q_i} \Big((\bar{D}_i \varphi)_{N_i+1} |D_i v|^2_{N_i + \frac{1}{2}} - (\bar{D}_i \varphi)_0 |D_i v|^2_{\frac{1}{2}} \Big) \, dt.$$

We observe that

$$\nu(h, \lambda) = \frac{h_i}{2} \iint_{Q_i} \Big((\varphi \, |D_i v|^2)_{\frac{1}{2}} + (\varphi \, |D_i v|^2)_{N_i + \frac{1}{2}} \Big) dt$$
$$+ \frac{h_i^2}{4} \iint_{Q_i} \Big((\bar{D}_i \varphi)_{N_i+1} |D_i v|^2_{N_i + \frac{1}{2}} - (\bar{D}_i \varphi)_0 |D_i v|^2_{\frac{1}{2}} \Big) \, dt,$$

can be made non-negative for h sufficiently small once λ is fixed as $\bar{D}_i \varphi = \mathcal{O}_\lambda(1)$.

We have

$$\widetilde{\varphi}^i = \varphi + h^2 \mathcal{O}_\lambda(1), \quad \overline{\varphi}^i = \varphi + h^2 \mathcal{O}_\lambda(1),$$
$$D_j D_i \varphi = \mathcal{O}_\lambda(1), \quad i, j \in [\![1, d]\!], j \neq i, \quad \bar{D}_i D_i \varphi = \mathcal{O}_\lambda(1), \; D_i \bar{D}_i \varphi = \mathcal{O}_\lambda(1), \; i[\![1, d]\!].$$

The result follows. ∎

Appendix D. A fully-discrete elliptic Carleman estimate for uniform meshes.

In Section 3 we have derived a Carleman estimate for a semi-discrete elliptic operator having in mind applications to the controllability of semi-discrete and discrete parabolic equations. For completeness, in the present section we treat the case of fully discrete elliptic operator. Here we thus only consider variables in $\Omega \subset \mathbb{R}^d$. The operator we consider is $A^{\mathfrak{M}} = -\sum_{i\in[\![1,d]\!]} \xi_{1,i}\bar{D}_i\xi_{2,i}D_i$. The case of a non uniform mesh can be treated as in Section 4.

We choose here to treat the case of an inner-observation in $\omega \Subset \Omega$. The weight function we choose is different from that introduced in Section 3. It is of the form $r = e^{s\varphi}$ with $\varphi = e^{\lambda\psi}$, with ψ fulfilling the following assumption. Construction of such a weight function is classical (see e.g. [FI96]).

Assumption D.1. *Let $\omega_0 \Subset \omega$ be an open set. Let $\tilde{\Omega}$ be a smooth open and connected neighborhood of $\overline{\Omega}$ in $\mathbb{R}^d$. The function $\psi = \psi(x)$ is in $\mathscr{C}^p(\overline{\tilde{\Omega}}, \mathbb{R})$, p sufficiently large, and satisfies, for some $c > 0$,*

$$\psi > 0 \text{ in } \tilde{\Omega}, \quad |\nabla\psi| \geq c \text{ in } \tilde{\Omega}\setminus\omega_0, \quad \partial_{n_i}\psi(t,x) \leq -c < 0 \text{ in } (0,\mathcal{T})\times V_{\partial_i\Omega},$$
$$\partial_i^2\psi(\boldsymbol{x}) \geq 0 \text{ in } V_{\partial_i\Omega}.$$

where $V_{\partial_i\Omega}$ is a sufficiently small neighborhood of $\partial_i\Omega$ in $\tilde{\Omega}$, in which the outward unit normal n_i to Ω is extended from $\partial_i\Omega$. We also set $\rho = r^{-1}$.

The following notation is adapted to the fully-discrete setting of the present section

$$\nabla_\xi f = \Big(\sqrt{\xi_{1,1}\xi_{2,1}}\partial_{x_1} f, \dots, \sqrt{\xi_{1,d}\xi_{2,d}}\partial_{x_d} f\Big)^t, \qquad \Delta_\xi f = \sum_{i\in[\![1,d]\!]} \xi_{1,i}\xi_{2,i}\partial_{x_i}^2 f.$$

As in Section 3 we use $\operatorname{reg}(\xi)$ to measure the boundedness of $\xi_{1,i}$ and $\xi_{2,i}$ and of their discrete derivatives as well as the distance to zero of $\xi_{1,i}$ and $\xi_{2,i}$, $i \in [\![1,d]\!]$ (see (3.1)-(3.2)). Here, by abuse of notation, the letters $\xi_{1,i}, \xi_{2,i}$ will also refer to a $\mathbb{Q}^1$-interpolation on $\mathfrak{M}$ and $\overline{\mathfrak{M}}^i$ respectively. Note that the resulting interpolated functions are Lipschitz continuous with

$$\|\xi_{1,i}\|_{W^{1,\infty}} \leq C\operatorname{reg}(\xi), \quad \|\xi_{2,i}\|_{W^{1,\infty}} \leq C\operatorname{reg}(\xi).$$

The enlarged neighborhood $\tilde{\Omega}$ of Ω introduced in Assumption 1.3 allows us to apply multiple discrete operators such as D_i and A_i on the weight functions. In particular, this then yields on $\partial_i\Omega$

$$(r\overline{D_i\rho}^i)_{|k_i=0} \leq 0, \quad (r\overline{D_i\rho}^i)_{|k_i=N_i+1} \geq 0, \quad i \in [\![1,d]\!].$$

Theorem D.2. *Let $\operatorname{reg}^0 > 0$ be given. For the parameter $\lambda \geq 1$ sufficiently large, there exist C, $s_0 \geq 1$, $h_0 > 0$, $\varepsilon_0 > 0$, depending on ω and reg^0, such that for any $\xi_{1,i}$, $\xi_{2,i}$, $i \in [\![1,d]\!]$, with $\operatorname{reg}(\xi) \leq \operatorname{reg}^0$ we have*

$$s^3\|e^{s\varphi}u\|^2_{L^2(\Omega)} + s\sum_{i\in[\![1,d]\!]}\|e^{s\varphi}D_iu\|^2_{L^2(\Omega)} + s\sum_{i\in[\![1,d]\!]}|e^{s\varphi}D_iu|^2_{L^2(\partial_i\Omega)}$$
$$\leq C_{\lambda_1,\mathfrak{K},\varepsilon_0,s_0}\Big(\|e^{s\varphi}A^{\mathfrak{M}}u\|^2_{L^2(\Omega)} + s^3\|e^{s\varphi}u\|^2_{L^2(\omega)}\Big).$$

for all $s \geq s_0$, $0 < h \leq h_0$ and $sh \leq \varepsilon_0$, and $u \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$, satisfying $u_{|\partial\Omega} = 0$.

Proof. We set $f := -A^{\mathfrak{M}} u$ and $v = ru$ that satisfies

$$r \sum_{i \in [\![1,d]\!]} \xi_{1,i} \bar{D}_i \xi_{2,i} D_i(\rho v) = rf.$$

Arguing as in the proof of Theorem 3.1 we then write $Av + Bv = g$ with $A = A_1 + A_2$ and $B = B_1 + B_2$ and

$$\begin{aligned}
A_1 v &= \sum_{i \in [\![1,d]\!]} \xi_{1,i} r \overline{\tilde{\rho}}^{i}\, \bar{D}_i(\xi_{2,i} D_i v), \quad A_2 v = \sum_{i \in [\![1,d]\!]} \xi_{1,i}\xi_{2,i} r (\bar{D}_i D_i \rho)\, \overline{\tilde{v}}^{i}, \\
B_1 v &= 2 \sum_{i \in [\![1,d]\!]} \xi_{1,i}\xi_{2,i} r \overline{D_i \rho}^{i}\, \overline{D_i v}^{i}, \quad B_2 v = -2s(\Delta_\xi \varphi) v \\
g &= rf - \sum_{i \in [\![1,d]\!]} \frac{h_i}{4} \xi_{1,i} r \overline{D_i \rho}^{i} (\bar{D}_i \xi_{2,i})(\tau_i^+ D_i v - \tau_i^- D_i v) \\
&\quad - \sum_{i \in [\![1,d]\!]} \frac{h_i^2}{4} \xi_{1,i} (\bar{D}_i \xi_{2,i}) r (\bar{D}_i D_i \rho) \overline{D_i v}^{i} - h_i \sum_{i \in [\![1,d]\!]} \mathcal{O}(1) r \overline{D_i \rho}^{i}\, \overline{D_i v}^{i} \\
&\quad - \sum_{i \in [\![1,d]\!]} \xi_{1,i} \Big(r(\bar{D}_i \xi_{2,i}) \overline{D_i \rho}^{i} + h_i \mathcal{O}(1) r (\bar{D}_i D_i \rho) \Big) \overline{\tilde{v}}^{i} - 2s(\Delta_{t,x}\varphi) v.
\end{aligned}$$

The proof of Lemma 3.2 can be directly adapted and we have

$$\|g\|^2_{L^2(\Omega)} \leq C_{\lambda,\mathfrak{K}} \left(\|rf\|^2_{L^2(\Omega)} + s^2 \|v\|^2_{L^2(\Omega)} + (sh)^2 \sum_{i \in [\![1,d]\!]} \|D_i v\|^2_{L^2(\Omega)} \right). \tag{D.1}$$

Developing the inner-product $\operatorname{Re}(Av, Bv)_{L^2(\Omega)}$, we set $I_{ij} = \operatorname{Re}(A_i v, B_j v)_{L^2(\Omega)}$.

LEMMA D.3 (Estimate of I_{11}). *For $sh \leq \mathfrak{K}$, the term I_{11} can be estimated from below in the following way*

$$I_{11} \geq -s\lambda^2 \|\varphi^{\frac{1}{2}} |\nabla_\xi \psi| \Upsilon_\xi v\|^2_{L^2(\Omega)} + Y_{11} - X_{11} - W_{11} - J_{11},$$

with

$$\begin{aligned}
Y_{11} = \sum_{i \in [\![1,d]\!]} \int_{\Omega_i} \Big(&\big((\xi_{1,i}^2 \xi_{2,i}^2 + \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2))\, r \overline{D_i \rho}^{i} \big)_{N_i+1} |D_i v|^2_{N_i + \frac{1}{2}} \\
&- \big((\xi_{1,i}^2 \xi_{2,i}^2 + \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2))\, r \overline{D_i \rho}^{i} \big)_0 |D_i v|^2_{\frac{1}{2}} \Big),
\end{aligned}$$

and

$$X_{11} = \sum_{i \in [\![1,d]\!]} \iint_\Omega \nu_{11,i} |D_i v|^2 + \sum_{i \in [\![1,d]\!]} \iint_\Omega \overline{\nu}_{11,i} |\overline{D_i v}^{i}|^2,$$

with $\nu_{11,i}$ and $\overline{\nu}_{11,i}$ of the form $s\lambda\varphi \mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ and

$$W_{11} = \sum_{\substack{i,j \in [\![1,d]\!] \\ i \neq j}} \iint_\Omega \gamma_{11,ij} |D_i D_j v|^2 + \sum_{i \in [\![1,d]\!]} \iint_\Omega \gamma_{11,ii} |\bar{D}_i D_i v|^2,$$

with $\gamma_{11,ij}$ and $\gamma_{11,ii}$ of the form $h^2 \big(s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) \big)$ and

$$J_{11} = \sum_{i \in [\![1,d]\!]} \int_{\Omega_i} \Big((\delta^{(2)}_{11,i})_{N_i + \frac{1}{2}} |D_i v|^2_{N_i + \frac{1}{2}} + (\delta^{(2)}_{11,i})_{\frac{1}{2}} |D_i v|^2_{\frac{1}{2}} \Big),$$

with $\delta^{(2)}_{11,i} = sh_i\lambda\varphi\mathcal{O}(1) + sh_i\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$. For a proof, see the proof of Lemma 3.3 in Appendix C and only consider the terms Q_{ii} and Q_{ij}.

LEMMA D.4 (Estimate of I_{12}). *For $sh \leq \mathfrak{K}$, the term I_{12} is of the following form*

$$I_{12} \geq 2s\lambda^2 \|\varphi^{\frac{1}{2}} |\nabla_\xi \psi| \Upsilon_\xi v\|^2_{L^2(\Omega)} - X_{12},$$

with

$$X_{12} = \sum_{i\in[\![1,d]\!]} \iint_\Omega \nu_{12,i} |D_i v|^2 + \iint_\Omega \mu_{12}|v|^2,$$

where $\mu_{12} = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)$, *and* $\nu_{12,i} = s\lambda\varphi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$.

LEMMA D.5 (Estimate of I_{21}). *For $sh \leq \mathfrak{K}$, the term I_{21} can be estimated from below in the following way*

$$I_{21} \geq 3s^3\lambda^4 \|\varphi^{\frac{3}{2}} |\nabla_\xi \psi|^2 v\|^2_{L^2(\Omega)} + Y_{21} - W_{21} - X_{21},$$

with

$$\begin{aligned}
Y_{21} = &\sum_{i\in[\![1,d]\!]} \int_{\Omega_i} \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)(r\overline{D_i\rho}^i)_0 |D_i v_{|\frac{1}{2}}|^2 \\
&+ \sum_{i\in[\![1,d]\!]} \int_{\Omega_i} \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)(r\overline{D_i\rho}^i)_{N_x+1} |D_i v_{|N_x+\frac{1}{2}}|^2, \\
W_{21} = &\sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \iint_\Omega \gamma_{21,ij} |D_i D_j v|^2, \qquad X_{21} = \iint_\Omega \mu_{21}|v|^2 + \sum_{i\in[\![1,d]\!]} \iint_\Omega \nu_{21,i}|D_i v|^2,
\end{aligned}$$

where

$$\begin{aligned}
&\gamma_{21,ij} = h\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2), \quad \mu_{21} = (s\lambda\varphi)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
&\nu_{21,i} = s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2).
\end{aligned}$$

For a proof, adapt the proof of Lemma 3.5 in Appendix C as was done for Lemma D.3.

LEMMA D.6 (Estimate of I_{22}). *For $sh \leq \mathfrak{K}$, the term I_{22} is of the following form*

$$I_{22} = -2s^3\lambda^4 \|\varphi^{\frac{3}{2}} |\nabla_\xi \psi|^2 v\|^2_{L^2(\Omega)} - X_{22},$$

with

$$X_{22} = \iiint_\Omega \mu_{22}|v|^2 + \sum_{i\in[\![1,d]\!]} \iiint_\Omega \nu_{22,i}|D_i v|^2$$

where $\mu_{22} = (s\lambda\varphi)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$, *and* $\nu_{22,i} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$.

With the previous lemmata, arguing as in the proof of Theorem 3.1, using that

$$(r\overline{D_i\rho}^i)_{N_i+1} \geq c > 0 \text{ and } -(r\overline{D_i\rho}^i)_0 \geq c > 0 \text{ on } \Omega_i, \quad 1 \leq i \leq d$$

by Assumption D.1 since $r\overline{D_i\rho}^i = -s\lambda(\partial_i\psi)\varphi + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$, we obtain that for some $\lambda_1 \geq 1$ sufficiently large, $s_1(\lambda_1) > 1$, $h_1(\lambda) > 0$ and $\varepsilon_1(\lambda_1) > 0$ then for $\lambda = \lambda_1$ (fixed for the rest of the proof), $s \geq s_1(\lambda_1)$, $0 < h \leq h_1(\lambda_1)$ and $sh \leq \varepsilon_1(\lambda_1)$ we have

$$\begin{aligned}
s^3\|v\|^2_{L^2(\Omega)} &+ s\sum_{i\in[\![1,d]\!]} \|D_i v\|^2_{L^2(\Omega)} + s\sum_{i\in[\![1,d]\!]} |D_i v|^2_{L^2(\partial_i\Omega)} \\
&\leq C_{\lambda_1,\mathfrak{K},\varepsilon_0,s_0}\Big(\|rf\|^2_{L^2(\Omega)} + s^3\|v\|^2_{L^2(\omega_0)} + s\sum_{i\in[\![1,d]\!]} \|D_i v\|^2_{L^2(\omega_0)}\Big).
\end{aligned}$$

Observe that the terms

$$s^3\|v\|^2_{L^2(\omega_0)} + s \sum_{i\in[\![1,d]\!]} \|D_i v\|^2_{L^2(\omega_0)}$$

in the r.h.s. appear as we only have $|\nabla\psi| \geq c > 0$ in $\Omega\backslash\omega_0$. Adding these two terms on the both sides of the estimate allows us to then proceed as in Section 3. In particular We can then use a result similar to that of Lemma 3.8.

Proceeding as in the end of proof of Theorem 4.1 in [BHL09a] we obtain

$$\begin{aligned} s^3\|e^{s\varphi}u\|^2_{L^2(\Omega)} &+ s\sum_{i\in[\![1,d]\!]}\|e^{s\varphi}D_iu\|^2_{L^2(\Omega)} + s\sum_{i\in[\![1,d]\!]}|e^{s\varphi}D_iu|^2_{L^2(\partial_i\Omega)} \\ &\leq C'_{\lambda_1,\mathfrak{K},\varepsilon_0,s_0}\Big(\|e^{s\varphi}f\|^2_{L^2(\Omega)} + s^3\|e^{s\varphi}u\|^2_{L^2(\omega_0)} + s\sum_{i\in[\![1,d]\!]}\|e^{s\varphi}D_iu\|^2_{L^2(\omega_0)}\Big). \end{aligned}$$

It thus remains to eliminate the last term in the r.h.s.. To that purpose we adapt the procedure followed in the continuous case (see e.g. [FI96, FCG06, LL09]). We multiply the equation satisfied by u, *i.e.* $A^{\mathfrak{M}}u = f$, by $sr^2\chi u^*$, where $\chi \in \mathscr{C}^\infty_c(\omega)$ is such that $\chi \geq 0$ and $\chi = 1$ in a neighborhood of ω_0. We then integrate over Ω:

$$-\operatorname{Re} s\sum_{i\in[\![1,d]\!]}\iint_\Omega \xi_{1,i}r^2\chi u^*\bar{D}_i(\xi_{2,i}D_iu) = \operatorname{Re} s\iint_\Omega r^2\chi u^* f. \tag{D.2}$$

We first note that the r.h.s. can be estimated by

$$\Big|\operatorname{Re} s\iint_\Omega r^2\chi u^* f\Big| \leq C\|rf\|^2_{L^2(\Omega)} + s^2C\|ru\|^2_{L^2(\omega)}. \tag{D.3}$$

In the l.h.s. of (D.2) we perform a discrete integration by parts to yield

$$\begin{aligned} -\operatorname{Re} s\sum_{i\in[\![1,d]\!]}\iint_\Omega \xi_{1,i}r^2\chi u^*\bar{D}_i(\xi_{2,i}D_iu) &= \operatorname{Re} s\sum_{i\in[\![1,d]\!]}\iint_\Omega D_i(\xi_{1,i}r^2\chi u^*)\xi_{2,i}D_iu \\ &= s\sum_{i\in[\![1,d]\!]}\iint_\Omega \xi_{2,i}\widetilde{\xi_{1,i}r^2\chi}^{\,i}|D_iu|^2 + \operatorname{Re} s\sum_{i\in[\![1,d]\!]}\iint_\Omega \xi_{2,i}D_i(\xi_{1,i}r^2\chi)\widetilde{u^*}^{\,i}D_iu \end{aligned} \tag{D.4}$$

In ω_0, for h sufficiently small, we have

$$\widetilde{\xi_{1,i}r^2\chi}^{\,i} \geq \widetilde{\xi_{1,i}r^2}^{\,i} = \widetilde{\xi_{1,i}}^{\,i}\,\widetilde{r^2}^{\,i} + \frac{h_i^2}{4}(D_i\xi_{1,i})(D_ir^2).$$

The results of the lemmata of Section 2.2 remain valid for r^2 in place of r, *i.e.* for s changed into $2s$. As $\widetilde{\xi_{1,i}}^{\,i} = \xi_{1,i} + h\mathcal{O}(1)$ and $D_i\xi_{1,i} = \mathcal{O}(1)$ we thus find

$$\widetilde{\xi_{1,i}r^2\chi}^{\,i} \geq r^2(\xi_{1,i} + h\mathcal{O}(1) + \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)).$$

For the first term in the r.h.s. of (D.4) it follows that, for h and sh sufficiently small,

$$s\sum_{i\in[\![1,d]\!]}\iint_\Omega \xi_{2,i}\widetilde{\xi_{1,i}r^2\chi}^{\,i}|D_iu|^2 \geq Cs\sum_{i\in[\![1,d]\!]}\|rD_iu\|^2_{L^2(\omega_0)}. \tag{D.5}$$

For second term in the r.h.s. of (D.4) we write

$$\begin{aligned} &\operatorname{Re} s\sum_{i\in[\![1,d]\!]}\iint_\Omega \xi_{2,i}D_i(\xi_{1,i}r^2\chi)\,\widetilde{u^*}^{\,i}D_iu \\ &= \operatorname{Re} s\sum_{i\in[\![1,d]\!]}\iint_\Omega \xi_{2,i}\widetilde{r^2}^{\,i}D_i(\xi_{1,i}\chi)\,\widetilde{u^*}^{\,i}D_iu + \frac{1}{2}s\sum_{i\in[\![1,d]\!]}\iint_\Omega \xi_{2,i}D_i(r^2)\widetilde{\xi_{1,i}\chi}^{\,i}D_i|u|^2. \end{aligned} \tag{D.6}$$

Arguing as above, the first term in the r.h.s. of (D.6) can be estimated by

$$\Big| \operatorname{Re} s \iint_\Omega \xi_{2,i} \widetilde{r^2}^i D_i(\xi_{1,i}\chi)\, \widetilde{u^*}^i D_i u \Big| \le C\|rD_i u\|^2_{L^2(\Omega)} + Cs^2 \|ru\|^2_{L^2(\omega)}, \tag{D.7}$$

for h and sh sufficiently small, as $\operatorname{supp}(\chi) \Subset \omega$. For the second term in the r.h.s. of (D.6) a discrete integration by parts yields

$$\frac{1}{2} s \sum_{i \in [\![1,d]\!]} \iint_\Omega \xi_{2,i} D_i(r^2) \widetilde{\xi_{1,i}\chi}^i D_i |u|^2 = -\frac{1}{2} s \sum_{i \in [\![1,d]\!]} \iint_\Omega D_i(\xi_{2,i} D_i(r^2) \widetilde{\xi_{1,i}\chi}^i) |u|^2$$

With the results of Section 2.2, using that $D_i \xi_{1,i} = \mathcal{O}(1)$ and $D_i \xi_{2,i} = \mathcal{O}(1)$ we find

$$\Big| \frac{1}{2} s \sum_{i \in [\![1,d]\!]} \iint_\Omega \xi_{2,i} D_i(r^2) \widetilde{\xi_{1,i}\chi}^i D_i |u|^2 \Big| \le Cs^3 \|ru\|^2_{L^2(\omega)}, \tag{D.8}$$

for h and sh sufficiently small.

With (D.2)–(D.8) we conclude that

$$s \sum_{i \in [\![1,d]\!]} \|rD_i u\|^2_{L^2(\omega_0)} \le C\Big(\|rf\|^2_{L^2(\Omega)} + s^3 \|ru\|^2_{L^2(\omega)} + \sum_{i \in [\![1,d]\!]} \|rD_i u\|^2_{L^2(\Omega)} \Big).$$

For s sufficiently large we thus obtain the desired Carleman estimate. $\square$

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